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COMPARISON OF THE KALMAN FILTER AND EXPONENTIAL SMOOTHING TECHNIQUES OF FORECASTING UNITED STATES MARINE CORPS LOSSES IN THE REPUBLIC OF VIETNAM

William Thomas Allison

Naval Postgraduate School
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THESIS

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Comparison of the Kalman Filter
and Exponential Smoothing Techniques
of Forecasting United States Marine Corps Losses
in the Republic of Vietnam

by

William Thomas Allison
Major, United States Army
B.A., The University of Texas, 1960

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Author William T. Allison

Approved by: Eamon B. Barrett
Thesis Advisor

Jack R. Borsting
Chairman, Department of Operations Analysis

W. F. Koehler for R. F. Rinehart
Academic Dean

ABSTRACT

This paper investigates the application of the Kalman Filter and the General Exponential Smoothing techniques of forecasting. Both methods are derived and the similarities and differences between them are discussed. The two techniques are then applied to the practical problem of predicting weekly losses suffered by the U. S. Marine Corps units in the I Corps Tactical Zone in the Republic of Vietnam. The mean absolute error of the prediction is used as the criterion for choosing the better of the two methods. Results are given for both techniques as well as for the method of linear regression. In general the Kalman Filter provides the smallest mean absolute error for the three mathematical models; linear, growing sine with harmonics and frequency of sixteen, thirty-two, and fifty-two weeks, and a constant model.

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I. INTRODUCTION

One of the problems in any forecasting technique is what to do about additional data. That is, given a forecast based upon some data, how does one go about updating this forecast when additional data is obtained? Two such methods are discussed in some detail. These are the methods known as the Kalman Filter, named for R. E. Kalman, and General Exponential Smoothing, a name coined by Robert G. Brown.

Both the Kalman Filter and General Exponential Smoothing are derived and comparisons made for any similarities and differences. Both techniques are then applied to the problem of forecasting losses incurred by the U. S. Marine Corps operating in the I Corps Tactical Zone of the Republic of Vietnam. The actual losses are expressed in weeks and cover the period from December 24, 1965, to April 6, 1968. Losses include all Marine Corps personnel who die as a result of battle as well as non-battle causes. It also includes those personnel who must be evacuated from Vietnam because of wounds or injuries received from any cause.

The basis for comparing the Kalman Filter and General Exponential Smoothing lies in their similarities. In each case, a mathematical model is assumed. The model consists of coefficients, which are unkown, and functions, called fitting functions, whose values are known. The problem becomes one of estimating the coefficients, based upon previous data, then using this estimate with the fitting functions to obtain a forecast. The forecast error, the difference between what actually occurs and what has been forecasted to occur, is weighed to obtain a new estimate of the coefficients.

A. THE MODELS

The models used for forecasting U. S. Marine losses are a linear model,

$$\hat{C}(t) = a_1 + a_2 t,$$

a growing sinusoidal with harmonics,

$$\hat{C}(t) = (a_1 + a_2 t) + (a_3 + a_5 t)\sin\frac{2\pi t}{w} + (a_4 + a_6 t)\cos\frac{2\pi t}{w} + a_7 \sin\frac{4\pi t}{w} + a_8 \cos\frac{4\pi t}{w},$$

and a constant model,

$$\hat{C}(t) = a_1 .$$

In the second model, the forecasts were made for values of w of sixteen, thirty-two, and fifty-two weeks.

The second model was chosen since there appeared to be some cycle in the number of losses incurred. It was felt that losses would increase as a result of increased enemy activity. Such activity was at a low ebb during the time of the monsoon rains. During the dry period, the tempo of enemy offensive actions increased for a time with a resultant increase in Marine Corps losses.

B. NOTATION

The notation used above reflects that used throughout this paper. The symbols $\hat{\theta}(t)$ or $\hat{C}(t)$ represent the forecast of losses for the t^{th} time period. The same symbols without the ^ above them are the observed value or the actual losses during the t^{th} period. It is also convenient to adopt vector and matrix notation. Thus, an alphabetical symbol, such as $\underset{\sim}{a}$, will represent a column vector, while a^T will denote a row vector.

In general, a vector with a subscript will represent the vector at a specific point in time. Matrices will be denoted by a capital alphabetical character. In the event the matrix symbol is subscripted, it will denote the matrix obtained by taking the expected value of a column vector, the first subscript, multiplied by the transpose of the vector denoted by the second subscript. A matrix with a single subscript is the expected value of the vector represented by the subscript times its transpose. For example,

$$C_{ab} = E(ab^T), \text{ and}$$

$$C_a = E(aa^T).$$

II. THE KALMAN FILTER

Basic to every forecasting technique is the criterion which is used to determine the forecast. In the case of the Kalman Filter, which estimates the coefficients of the model, the criterion is to minimize the trace of the estimation error covariance matrix, C_e, where e is the unobservable error vector between the true value of the model coefficients and the current estimate of these coefficients. This estimate is the minimum mean square error estimator.

A fundamental theorem of estimation, called the Gauss-Markov Theorem, is very important in any form of estimation. The theorem shall be stated but no proof is provided.

Theorem: If θ is the vector of the observed data at times t_i and $\hat{x}$ is a linear function of θ, x and θ are random variables with moment matrices C_x, $C_{x\theta}$, and C , and if C^{-1} exists, then the linear minimum mean square estimate $\hat{x}$ of x given θ is

$$\hat{x} = C_{x\theta}C^{-1}\theta ,$$

and $$C_e = C_x - C_{x\theta}C_\theta^{-1}C_{x\theta}^T .$$

Throughout this paper, the data will be assumed to be linearly related to the vector x (or a). Thus,

$$\theta = Bx + v,$$

where v is the measurement error or noise vector and B is an $m \times n$ matrix whose rows are the vectors $f^T(t)$, the vector of fitting functions evaluated at time t. For example, if $\theta(t) = x_1 + x_2t + v$, the vector of fitting functions $f^T(t) = (1,t)$ and the vector of coefficients $x^T = (x_1 , x_2)$.

Then,

$$C_{x\theta} = E[x(Bx + v)^T]$$

or $$C_{x\theta} = C_xB^T + C_{xv}$$

and $$C_\theta = E[Bx + v)(Bx + v)^T],$$

or $$C_\theta = BC_xB^T + BC_{xv} + C_{xv}^TB^T + C_v .$$

From the fundamental theorem,

$$\hat{x} = [C_xB^T + C_{xv}][BC_xB^T + BC_{xv} + C_{xv}^T + C_v]^{-1}\theta ,$$

and $$C_e = C_x - [C_xB^T + C_{xv}][BC_xB^T + C_{xv}^TB^T + BC_{xv} + C_v]^{-1}[C_xB^T + C_{xv}]^T.$$

Normally the vectors x and v are assumed to be independent. This, with the assumption that $E(v) = 0$, implies that C_{xv} is identically zero. With this assumption, the above equations reduce to

$$\hat{x} = C_xB^T[BC_xB^T + C_v]\theta$$

and $$C_e = C_x - C_x B^T [B C_x B^T + C_v]^{-1} [B C_x^T]^T .$$

By the use of the matrix identities

$$(2.1) \quad AB^T[C + BAB^T]^{-1} = [A^{-1} + B^T C^{-1} B]^{-1} B^T C^{-1},$$

and $$(2.2) \quad [C^{-1} + B^T A^{-1} B]^{-1} = C - CB^T[BCB^T + A]^{-1} BC,$$

the following equations are obtained:

$$\hat{\underset{\sim}{x}} = [C_x^{-1} + B^T C_v^{-1} B]^{-1} B^T C_v^{-1} \underset{\sim}{\theta}$$

and $$C_e = [C_x^{-1} + B^T C_v^{-1} B]^{-1}.$$

There still remains the matrix C_x which must be known before the estimate $\hat{\underset{\sim}{x}}$ can be made. In general little information is available concerning $\underset{\sim}{x}$. In this case it is reasonable to assume that the diagonal elements of C_x are larger than any finite number. With this assumption, C_x^{-1} is considered to be 0. Then,

$$\hat{\underset{\sim}{x}} = [B^T C_v^{-1} B]^{-1} B^T C_v^{-1} \underset{\sim}{\theta},$$

and $$C_e = [B^T C_v^{-1} B]^{-1}.$$

A. THE KALMAN FILTER EQUATIONS

The foregoing has been the groundwork. Suppose that m observations, $\underset{\sim}{\theta}^m$, have been taken and $\hat{\underset{\sim}{x}}^m$ is the minimum mean square error estimate based upon the m observations. The matrix C_e^m is available. In addition, suppose that r additional observations are taken. The problem is to determine the estimate $\hat{\underset{\sim}{x}}^{r+m}$.

Since $\hat{\underset{\sim}{x}}^m$ is available and is

$$\hat{\underset{\sim}{x}}^m = [B^{mT}(C_v^m)^{-1} B^m]^{-1} B^{mT} (C_v^m)^{-1} \underset{\sim}{\theta}^m$$

$$C_e^m = [B^{mT}(C_v^m)^{-1}B^m]^{-1} .$$

Also, $$\underset{\sim}{\theta}^m = B^m\underset{\sim}{x}^m + \underset{\sim}{v}^m ,$$

and $$\underset{\sim}{\theta}^{m+r} = B^{m+r}\,\underset{\sim}{x}^{m+r} + \underset{\sim}{v}^{m+r} .$$

The noise vector $\underset{\sim}{v}^m$ is independent of the vector $\underset{\sim}{v}^r$. This implies that C_v^{m+r} is a diagonal matrix with elements C_v^m and C_v^r. By partitioning the matrix B^{m+r} into components B^m and B^r and the vector $\underset{\sim}{\theta}^{m+r}$ into $\underset{\sim}{\theta}^m$ and $\underset{\sim}{\theta}^r$, the new estimate

$$\hat{\underset{\sim}{x}}^{m+r} = [(C_e^m)^{-1} + B^{rT}(C_v^r)^{-1}B^r]^{-1}[(C_e^m)^{-1}x^m + B^{rT}(C_v^r)^{-1}\underset{\sim}{\theta}^r]$$

is obtained. Also,

$$C_e^{m+r} = [(C_e^m)^{-1} + B^{rT}(C_v^r)^{-1}B^r]^{-1} .$$

Then define $\underset{\sim}{K} = C_e^{m+r}B^{rT}(C_v^r)^{-1}$. From equation (2.1),

$$\underset{\sim}{K} = C_e^mB^{rT}[C_v^r + B^rC_e^mB^{rT}]^{-1} .$$

From equation (2.2),

$$C_e^{m+r} = C_e^m - \underset{\sim}{K}B^rC_e^m .$$

Finally, the new estimate is

$$\hat{\underset{\sim}{x}}^{m+r} = \hat{\underset{\sim}{x}}^m + \underset{\sim}{K}[\underset{\sim}{\theta}^r - B^r\hat{\underset{\sim}{x}}^m] .$$

These last three equations are the Kalman Filter. [Liebelt 1967, pp. 165-166] Note that succeeding estimates $\hat{\underset{\sim}{x}}^{m+r}$ are computed recursively from the prior estimate $\hat{\underset{\sim}{x}}^m$.

B. REMARKS

Notice that with the assumption that C_v is known, neither $\underset{\sim}{K}$ nor C_e depend upon the data $\underset{\sim}{\theta}$. These elements could be precomputed and retained for later use as needed. Also, it is not necessary to assume that the matrix C_x^{-1} is zero. If one has information about the distribution of $\underset{\sim}{x}$, the elements of C_x can be computed and used in determining C_e.

The vector $\underset{\sim}{K}$ may be thought of as a weighting factor applied to the forecast error. Notice that the weight for any error is proportional to the inverse of the variance of the forecast. A large variance in the measurement or noise for any particular forecast will cause the error to be given less weight in determining the estimate of the coefficients to be used in computing the next forecast.

The dimension of the matrix C_v^r is always r x r. If measurements or observations are taken at each succeeding time period, C_v^1 is a scalar quantity and is the variance of the measurement for that period. In the event that the matrix C_v is diagonal with the diagonal elements equal to some constant raised to some power, the matrix C_e will reach a steady state. That is,

$$C_e^{m+r} = C_e^m \ .$$

The power to which the constant is raised corresponds to the time period for which the observation is made.

III. GENERAL EXPONENTIAL SMOOTHING

As is the case in the Kalman Filter method, the linear model

$$\hat{\underset{\sim}{x}} = A\underset{\sim}{\theta}$$

is assumed, but rather than estimating $\hat{\underset{\sim}{x}}$ in order to minimize the diagonal elements of some matrix, it is desired to minimize the sum of the weighted squared error,

$$\sum_{j=1}^{m} w_j^2[\theta(j) - \hat{\theta}(j)]^2 .$$

If W is a diagonal matrix with elements $(w_1,\ldots,w_m)$, then

$$\sum_{i=1}^{m} w_i^2[\theta(i) - \hat{\theta}(i)]^2 = [W(\underset{\sim}{\theta} - \hat{\underset{\sim}{\theta}})]^T W(\underset{\sim}{\theta} - \hat{\underset{\sim}{\theta}}) .$$

Also, it is assumed that

$$\hat{\underset{\sim}{\theta}} = B\hat{\underset{\sim}{x}} ,$$

where the matrix B is defined the same as in the discussion of the Kalman Filter.

Define the matrix F to be B^TWW^TB. Then

$$[W(\underset{\sim}{\theta} - \hat{\underset{\sim}{\theta}})]^T W(\underset{\sim}{\theta} - \hat{\underset{\sim}{\theta}}) = [\underset{\sim}{\theta}^T A^T - \underset{\sim}{\theta}^T WW^T BF^{-1}]F[\underset{\sim}{\theta}^T A^T - \underset{\sim}{\theta}^T WW^T BF^{-1}]^T$$

$$- \underset{\sim}{\theta}^T WW^T BF^{-1}[\underset{\sim}{\theta}^T WW^T B]^T + \underset{\sim}{\theta}^T W^T W\underset{\sim}{\theta} .$$

The matrix F is positive definite. Therefore the choice of A^T which minimizes the above expression is

$$A^T = WW^TBF^{-1} .$$

Finally,

$$\hat{\underset{\sim}{x}}^T = \underset{\sim}{\theta}^T WW^T BF^{-1} .$$

Notice that this is identical with the Gauss-Markov estimate with $F = B^T C_v^{-1} B$ and $WW^T = C_v^{-1}$.

A. THE WEIGHTING VECTOR

A fundamental hypothesis of Exponential Smoothing is that the vector of fitting functions f(t+1) must be a linear combination of f(t). The matrix of coefficients of this combination, which does not depend on time, is called the transition matrix, L.

From this,

$$\underset{\sim}{f}(t+1) = L\underset{\sim}{f}(t) ,$$

and $$\hat{\underset{\sim}{\theta}}(t+1) = \underset{\sim}{f}^T(1)L^T\hat{\underset{\sim}{x}}(t-1) ,$$

where $$\hat{\underset{\sim}{x}}(t) = (L^T)^t\hat{\underset{\sim}{x}}(0) .$$

The fitting functions are always computed at the number of time periods in the future for which the forecast is desired. This implies that the present period is always considered as the period t = 0. Then the problem becomes how to compute $\hat{\underset{\sim}{x}}(t)$ recursively from $\hat{\underset{\sim}{x}}(t-1)$ so that

$\sum_{i=0}^{t} b^i[\theta(t-i) - \hat{\underset{\sim}{x}}(t)\underset{\sim}{f}(-i)]^2$ is minimized.

Define the matrix

$$F(T) = \sum_{i=0}^{T} b^i \underset{\sim}{f}(-i)\underset{\sim}{f}^T(-i) ,$$

and the matrix F to be the limit of F(t) as t becomes very large. The matrix F will exist if each fitting function $f_i(t)$ is greater than the square root of b^{-t}.

Let the data vector be

$$\underset{\sim}{g}(T) = \sum_{i=0}^{T} b^i \theta(T-i)\underset{\sim}{f}(-i) .$$

Then the i^{th} element of the vector $\underset{\sim}{g}(T)$ is $\sum_{k=0}^{T} b^k \theta(T-k) f_i(-k)$.

Then $$\underset{\sim}{g}(T) = \theta(T)\underset{\sim}{f}(0) + bL^{-1}\underset{\sim}{g}(T-1) ,$$

and $$F(T)\hat{\underset{\sim}{x}}(T) = \sum_{i=0}^{T} b^i \underset{\sim}{f}(-i)\underset{\sim}{f}^T(-i)\hat{\underset{\sim}{x}}(T) .$$

But $$\theta(T-i) = \underset{\sim}{f}^T(-i)\hat{\underset{\sim}{x}}(T) .$$

Hence, $$F(T)\hat{\underset{\sim}{x}}(T) = \underset{\sim}{g}(T) .$$

Substituting the steady state matrix F and combining the two equation for $\underset{\sim}{g}(T)$, the expression

$$\hat{\underset{\sim}{x}}(T) = \theta(T)F^{-1}\underset{\sim}{f}(0) + bF^{-1}L^{-1}F\hat{\underset{\sim}{x}}(T-1)$$

is obtained. [Brown 1963].

But $$bF^{-1}L^{-1}F = L^T - F^{-1}\underset{\sim}{f}(0)\underset{\sim}{f}^T(0)L^T$$

and $$\underset{\sim}{f}^T(1) = \underset{\sim}{f}^T(0)L^T .$$

Define the vector $\underset{\sim}{h}$ to be $F^{-1}\underset{\sim}{f}(0)$. Then the expression for $\hat{\underset{\sim}{x}}(T)$ becomes

$$\hat{\underset{\sim}{x}}(T) = \underset{\sim}{h}\theta(T) + [L^T - \underset{\sim}{h}\underset{\sim}{f}^T(1)]\hat{\underset{\sim}{x}}(T-1) ,$$

or $$\hat{\underset{\sim}{x}}(T) = L^T\hat{\underset{\sim}{x}}(T-1) + \underset{\sim}{h}[\theta(T) - \hat{\theta}(T)] .$$

This last equation is the Exponential Smoothing means of estimating the model coefficients. [Brown 1963, pp. 174-177]. The vector $\underset{\sim}{h}$ is the weight of the forecast error applied in obtaining the new estimate $\hat{\underset{\sim}{x}}(T)$. Note that $\hat{\underset{\sim}{x}}(T)$ is computed in a recursive way from $\hat{\underset{\sim}{x}}(T-1)$.

B. THE FORECAST BIAS

One may question the accuracy of the forecast obtained by means of Exponential Smoothing. It has been shown [Bessler and Zehna 1968] that for the model

$$C(t) = a + e(t) ,$$

where

$$E[C(t)] = a$$

that the forecast is biased in the early stages by a factor of $1-b^t$, where b is the weighting scalar applied to past data. Since b is between zero and one, then $E[C(t)] = a$ when b^t has vanished. However, this shows that even where the mathematical model is correct, there is an initial bias in the forecast.

This may lead one to ponder what occurs when the assumed model is incorrect. It has been demonstrated [Brown 1963, p. 128] that in the case where the assumed model is

$$E[C(t)] = a$$

but the data is actually corresponding to the model

$$E[C(t)] = a_1 + a_2 t ,$$

that

$$E[\hat{C}(t)] = a_1 + a_2 t - a_2 b(1-b)^{-1} ,$$

after b^t vanishes. Thus, even after a sufficient amount of time has elapsed so that $b^t = 0$, the forecast is incorrect or biased by a constant times the slope of the line through the data.

The next step was to determine if these results could be generalized. To do this, the Exponential Smoothing equation

$$\hat{\underline{a}}(t) = L^T\hat{\underline{a}}(t-1) + \underline{h}[C(t) - \underline{f}^T(1)\hat{\underline{a}}(t-1)]$$

was examined. This can be rewritten as

$$\hat{a}(t) = \sum_{k=0}^{t-1} [L^T - hf^T(1)]^k hC(t-k) + [L^T - hf^T(1)]^t \hat{a}(0).$$

Then if $\qquad E[C(t)] = g^T(t)a$,

$$E[\hat{C}(t)] = f^T(1) \sum_{k=0}^{t-1} [L^T - hf^T(1)]^k hg^T(t-k)a + f^T(1)[L^T - hf^T(1)]^t E[\hat{a}(0)].$$

The above expression is true in general when the assumed model is

$$C(t) = f^T(t)a + e(t)$$

and the data actually follows the model

$$C(t) = g^T(t)a + e(t).$$

In order to compare this expression with those obtained previously for the constant model, the values

$$L = 1,$$

$$f(t) = 1,$$

and $\qquad g(t) = 1$

are substituted to obtain

$$E[\hat{C}(t)] = \sum_{k=0}^{t-1} [1-h]^k h\ a\ .$$

This can be written as

$$E[\hat{C}(t)] = a[1-b^t]\ ,$$

where b is 1-h. Thus the result is the same as before.

Next, it was desired to consider the situation where the data was modeled by $E[\hat{C}(t)] = a$, but the data actually obeyed the model

$$E[C(t) = a_1 + a_2t.$$

Then substituting the values

$$L^T = 1,$$

$$f^T(1) = 1,$$

$$g^T(t) = (1,t),$$

and $$\underset{\sim}{a}^T = (a_1, a_2)$$

into the expression for $\hat{E}[C(t)]$,

$$E[\hat{C}(t)] = \sum_{k=0}^{t-1} [1-h]^k(1,t)(a_1, a_2)^T h .$$

This becomes $$E[\hat{C}(t)] = (a_1+a_2t)(1-h^t) - a_2(1-h)h^{-1}(1-h^t) .$$

When h^t vanishes, the expression becomes

$$E[\hat{C}(t)] = a_1 + a_2t - a_2(1-h)h^{-1},$$

which is identical with the prior result with $b = 1-h$.

Notice that the expression for $E[\hat{C}(t)]$ has been used to determine if a bias exists when the correct mathematical model is assumed as well as to determine the amount of the bias when an incorrect model is assumed. It was desirable to determine if the forecast was biased when the assumed linear model was correct. The equation then becomes

$$\begin{aligned} E[\hat{C}(t)] = {} & (a_1+a_2t)\underset{\sim}{f}^T(1)[I-\underset{\sim\sim}{L}^T+h\underset{\sim}{f}^T(1)]^{-1}[I-(\underset{\sim\sim}{L}^T-h\underset{\sim}{f}^T(1))^t]\underset{\sim}{h} \\ & - a_2\underset{\sim}{f}^T(1)[I-\underset{\sim\sim}{L}^T+h\underset{\sim}{f}^T(1)]^{-1}[\underset{\sim\sim}{L}^T-h\underset{\sim}{f}^T(1)][I-\underset{\sim\sim}{L}^T+h\underset{\sim}{f}^T(1)]^{-1} \\ & [I-(\underset{\sim\sim}{L}^T-h\underset{\sim}{f}^T(1))^{t-1}]\underset{\sim}{h}-(t-1)[I-\underset{\sim\sim}{L}^T+h\underset{\sim}{f}^T(1)]^{-1}[\underset{\sim\sim}{L}^T-h\underset{\sim}{f}^T(1)]^{t-1}\underset{\sim}{h} . \end{aligned}$$

It became important to know if the vector $[L^T-hf^T(1)]^t h$ vanished as t became large. If the vector fails to vanish, the forecast will remain biased. The vector will vanish if the eigenvalues of $[L^T-hf^T(1)]$ are all between minus one and plus one. To demonstrate this, let $A = L^T-hf^T(1)$, where A is n x n and nonsingular. Also, let the eigenvalues of A be $m_1, m_2, \ldots, m_n$, where each m_i lies between minus one and plus one. Let $v_1, v_2, \ldots, v_n$ be eigenvectors of A. These vectors form a basis for E^n.

Then
$$Av_1 = m_1 v_1 ,$$
$$\vdots$$
$$Av_n = m_n v_n .$$

Also, if x is in E^n, then

$$x = a_1 v_1 + \ldots + a_n v_n .$$

Premultiplying each side by A,

$$Ax = a_1 Av_1 + \ldots + a_n Av_n.$$

Substitute to obtain

$$Ax = a_1 m_1 v_1 + \ldots + a_n m_n v_n.$$

Repeating this process t times,

$$A^t x = a_1 m_1^t v_1 + \ldots + a_n m_n^t v_n.$$

Each $m_i^t = 0$ for sufficiently large t. Therefore,

$$A^t x = 0.$$

Since this is true for all x in E^n, $A^t h = 0$.

In the case of the linear model, the eigenvalues are equal to $1 - (h_1 + h_2)2^{-1}$. These eigenvalues will be between minus one and plus one if $h_1 + h_2$ lies between zero and four. The $\underset{\sim}{h}$ for the linear model is such that this is true, if b is between zero and one. To demonstrate this, recall that

$$\underset{\sim}{h} = F^{-1}\underset{\sim}{f}(0),$$

$$F = \sum_{i=0}^{\infty} b^i \underset{\sim}{f}(-i)\underset{\sim}{f}^T(-i) ,$$

and

$$\underset{\sim}{f}^T(t) = (1, t) .$$

Then

$$F^{-1} = \begin{bmatrix} 1-b^2 & (1-b)^2 \\ (1-b)^2 & (1-b)^3 b^{-1} \end{bmatrix} .$$

Then $\underset{\sim}{h}^T = (h_1, h_2) = [1-b^2, (1-b)^2]$. Hence,

$$h_1 + h_2 = 2(1-b)^2,$$

which is less than four and is greater than zero.

Therefore, the vector $[L^T-\underset{\sim}{h}\underset{\sim}{f}^T(1)]^t\underset{\sim}{h}$ vanishes in the linear model for all values of b which are between zero and one. Then

$$\begin{aligned} E[\hat{C}(t)] &= (a_1+a_2t)\underset{\sim}{f}^T(1)[I-L^T+\underset{\sim}{h}\underset{\sim}{f}^T(1)]^{-1}\underset{\sim}{h} \\ &\quad - a_2\underset{\sim}{f}^T(1)[I-L^T+\underset{\sim}{h}\underset{\sim}{f}^T(1)]^{-1}[L^T-\underset{\sim}{h}\underset{\sim}{f}^T(1)][I-L^T+\underset{\sim}{h}\underset{\sim}{f}^T(1)]^{-1}\underset{\sim}{h} . \end{aligned}$$

Thus the forecast is unbiased if

$$\underset{\sim}{f}^T(1)[I - L^T + \underset{\sim}{h}\underset{\sim}{f}^T(1)]^{-1}\underset{\sim}{h} = 1$$

and

$$\underset{\sim}{f}^T(1)[I-L^T+\underset{\sim}{h}\underset{\sim}{f}^T(1)]^{-1}[L^T-\underset{\sim}{h}\underset{\sim}{f}^T(1)][I-L^T+\underset{\sim}{h}\underset{\sim}{f}^T(1)]^{-1}\underset{\sim}{h} = 0 .$$

For a specific example see Chapter V.

C. EXPONENTIAL SMOOTHING COMPARED WITH KALMAN FILTER

The Exponential Smoothing equation is a special case of the Kalman Filter in the steady state. Recall that the matrix F was defined to be B^TWW^TB. If WW^T is equivalent to the matrix C_v^{-1} used in the Kalman Filter equations, then F^{-1} is identical to the matrix C_e derived in Chapter II. Thus, in the case of fitting functions consisting of polynomials, exponentials, and sinusoidals, which can be generated by a transition matrix, the Kalman Filter estimate becomes identical with that of Exponential Smoothing when C_e reaches the steady state. However, since the Kalman Filter method does not require that the fitting functions be generated by means of a transition matrix, it may be applied to models other than those listed above.

The primary difference between the Kalman Filter and Exponential Smoothing is the models for which they are valid. The Kalman Filter is valid both in a dynamic system as well as in the steady state. The Exponential Smoothing technique is strictly valid only in the steady state.

IV. FORECASTING MARINE LOSSES

In order to see how the Kalman Filter and the Exponential Smoothing techniques compared, both methods, along with linear regression, were used to forecast U. S. Marine Corps losses in the Republic of Vietnam. The actual data used included one hundred and nineteen weeks from December 26, 1965, through April 6, 1968. The losses included those Marines who either were killed or were wounded and evacuated from Vietnam.

Three models were investigated. These were a growing sinusoidal with harmonics with a cycle of sixteen, thirty-two, and fifty-two weeks,

$$\hat{C}(t) = a_1+a_2t+(a_3+a_5t)\sin(pt) + (a_4+a_6t)\cos(pt) + a_7\sin(2pt) + a_8\cos(2pt) ,$$

a linear model,

$$\hat{C}(t) = a_1 + a_2t ,$$

and a constant model,

$$\hat{C}(t) = a .$$

The objective was to minimize the mean absolute error where the absolute error for the t^{th} forecast was the absolute value of $C(t) - \hat{C}(t)$. The mean error and the estimated variance of the errors were included for comparison.

Both the Kalman Filter and Exponential Smoothing require an estimate of the coefficients $\underset{\sim}{a}(0)$ in order to begin forecasting. Captain Paul William O'Brien, U. S. Marine Corps, had studied the problem of Exponential Smoothing in forecasting Marine monthly losses. [O'Brien 1968].

Using a least squares line through the monthly losses from March 1965 to December 1966 and the least squares Fourier fit through the deviations about the regression line, Captain O'Brien obtained initial values of the coefficients for the growing sinusoidal model. The coefficients of the least squares line through the data were used as the initial values in the linear model. These values were scaled by one-fourth and used as the initial values for forecasting weekly losses. Thus the growing sinusoidal model was initialized with

$$\underset{\sim}{a}^T(0) = (-36.45,\ 15.675,\ 62.61,\ 31.3975,\ .6325,\ .6325,\ -35.1175,\ 54.76)\ ,$$

and the linear model with

$$\underset{\sim}{a}^T(0) = (-36.45,\ 15.675)\ .$$

A. FORECASTING WITH EXPONENTIAL SMOOTHING

In each model the value of the weighting factor by which the errors were discounted was chosen to be the solution to

$$1 - b^n = .25\ ,$$

where n was the number of coefficients in the model. The weighting vector $\underset{\sim}{h}$ used in the Exponential Smoothing equation was computed by determining the steady state matrix F [Brown 1963]. Then

$$\underset{\sim}{h} = F^{-1}\,\underset{\sim}{f}(0)\ .$$

The transition matrix in each model was determined by inspection, remembering trignometric identities for sin(A+B) and cos(A+B).

The forecasting procedure involved several steps. These were:

Step 1; Initialize the vector $\underset{\sim}{a}(0)$ and compute $\underset{\sim}{f}(1)$.

Step 2; Compute the forecast $\hat{C}(1)$ by

$$\hat{C}(1) = \underset{\sim}{f}^T(1)\underset{\sim}{a}(0).$$

Step 3; Compute the forecast error

$$e(1) = C(1) - \hat{C}(1).$$

Step 4: Obtain a new estimate $\underset{\sim}{a}(1)$ by

$$\underset{\sim}{a}(1) = L^T\underset{\sim}{a}(0) + \underset{\sim}{h}e(1).$$

Step 5; Compute the forecast $\hat{C}(2)$ by

$$\hat{C}(2) = \underset{\sim}{f}^T(1)\underset{\sim}{a}(1).$$

Step 6; Repeat Steps 3 through 5 for each successive forecast forecast with the appropriate time arguments.

The first model investigated was the growing sinusoidal with harmonics. For this model the transition matrix is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos p & \sin p & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin p & \cos p & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos p & \sin p & \cos p & \sin p & 0 & 0 \\ 0 & 0 & -\sin p & \cos p & -\sin p & \cos p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos 2p & \sin 2p \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sin 2p & \cos 2p \end{bmatrix}$$

where p is 2π divided by the cycle length. Also

$$\underset{\sim}{f}^T(1) = (1,\ 1,\ \sin p,\ \cos p,\ \sin p,\ \cos p,\ \sin 2p,\ \cos 2p).$$

The values of $\underset{\sim}{h}$ used were, for the sixteen week cycle

$$\underset{\sim}{h}^T = (0.064709,\ 0.0011416,\ 0.025939,\ 0.12583,\ 0.00046034,\ 0.0022276,\ 0.020951,\ 0.059457),$$

for a thirty-two week cycle

$$\underset{\sim}{h}^T = (0.072061,\ 0.0012067,\ 0.056675,\ 0.12816,\ 0.00096984,\ 0.0021717,\ 0.042191,\ 0.049775),$$

and for a fifty-two week cycle

$$\underset{\sim}{h}^T = (0.089418,\ 0.0013569,\ 0.10333,\ 0.13127,\ 0.0016449,\ 0.0020314,\ 0.063796,\ 0.029308).$$

The results obtained for this model for the three cycles are contained in Tables I through III.

The second model was the linear model. The same procedure was followed as outlined above. The transition matrix was

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and $\underset{\sim}{f}^T(1) = (1, 1)$,

$$\underset{\sim}{h}^T = (0.25,\ 0.01795) .$$

The results from this model are contained in Table IV.

The last model to which Exponential Smoothing was applied was the constant model. With this model

$$L = 1,$$

$$f^T(1) = 1,$$

and $h = 0.25$.

TABLE I

GENERAL EXPONENTIAL SMOOTHING RESULTS
GROWING SINE WITH HARMONICS
CYCLE LENGTH 16 WEEKS

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
1	61	47	14
2	64	32	32
3	84	30	54
4	53	58	-5
5	87	89	-2
6	116	120	-4
7	119	128	-9
8	108	102	6
9	167	55	112
10	405	36	369
11	89	107	-18
12	85	107	-22
13	339	139	200
14	208	251	-43
15	80	294	-214
16	116	270	-154
17	141	245	-104
18	133	229	-96
19	125	227	-102
20	158	244	-86
21	164	282	-118
22	82	319	-237
23	203	318	-115
24	139	337	-198
25	133	321	-188
26	198	304	-106
27	151	315	-164
28	140	322	-182
29	175	332	-157
30	359	352	7
31	280	405	-125
32	1253	404	849
33	273	627	-354
34	150	491	-341
35	257	348	-91
36	559	307	252
37	258	402	-144
38	290	411	-121
39	382	418	-36
40	547	426	121
41	119	451	-332
42	125	353	-228
43	85	317	-232
44	54	341	-287
45	343	408	-65

TABLE I (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
46	118	556	-438
47	96	580	-484
48	80	541	-461
49	262	467	-205
50	131	447	-316
51	85	408	-323
52	177	395	-218
53	106	434	-328
54	435	444	-9
55	145	512	-367
56	91	439	-348
57	126	338	-212
58	442	274	168
59	86	337	-251
60	121	320	-199
61	153	340	-187
62	327	373	-46
63	191	429	-238
64	220	403	-183
65	336	367	-31
66	280	364	-84
67	237	357	-120
68	202	359	-157
69	300	369	-69
70	273	407	-134
71	389	412	-23
72	646	419	227
73	732	460	272
74	359	486	-127
75	400	397	3
76	253	362	-109
77	247	336	-89
78	249	350	-101
79	248	373	-125
80	536	376	160
81	249	426	-177
82	256	363	-107
83	254	324	-70
84	315	333	-18
85	208	406	-198
86	269	464	-195
87	203	525	-322
88	290	521	-231
89	608	495	113
90	357	517	-160
91	260	439	-179
92	189	360	-171
93	166	317	-151
94	262	321	-59
95	216	371	-155

TABLE I (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
96	224	385	-161
97	304	366	-62
98	493	343	150
99	234	364	-130
100	262	323	-61
101	466	338	128
102	367	440	-73
103	357	500	-143
104	414	520	-106
105	393	505	-112
106	243	444	-201
107	438	337	101
108	208	319	-111
109	404	277	127
110	948	333	615
111	735	529	206
112	1056	585	471
113	787	656	131
114	1339	590	749
115	453	667	-214
116	563	499	64
117	475	449	26
118	676	442	234
119	582	517	65

Mean Absolute Error = 170

Mean Error = -68.97

Error Variance = 4.42×10^4

TABLE II

GENERAL EXPONENTIAL SMOOTHING RESULTS
GROWING SINE WITH HARMONICS
CYCLE LENGTH 32 WEEKS

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
1	61	60	1
2	64	64	0
3	84	63	21
4	53	67	-14
5	87	65	22
6	116	77	39
7	119	101	18
8	108	125	-17
9	167	145	22
10	405	177	228
11	89	266	-177
12	85	241	-156
13	339	210	129
14	208	246	-38
15	80	231	-151
16	116	176	-60
17	141	140	1
18	133	120	13
19	125	108	17
20	158	106	52
21	164	125	39
22	82	153	-71
23	203	162	41
24	139	212	-73
25	133	240	-107
26	198	264	-66
27	151	301	-150
28	140	316	-176
29	175	321	-146
30	359	331	28
31	280	386	-106
32	1253	404	849
33	273	681	-408
34	150	619	-469
35	257	521	-264
36	559	466	93
37	258	505	-247
38	290	457	-167
39	382	433	-51
40	547	446	101
41	119	509	-390
42	125	444	-319
43	85	395	-310
44	54	346	-292
45	343	302	41

TABLE II (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
46	118	348	-230
47	96	324	-228
48	80	297	-217
49	262	270	-8
50	131	300	-169
51	85	290	-205
52	177	271	-94
53	106	287	-181
54	435	287	148
55	145	385	-240
56	91	389	-298
57	126	381	-255
58	442	388	54
59	86	485	-399
60	121	464	-343
61	153	454	-301
62	327	451	-124
63	191	492	-301
64	220	480	-260
65	336	471	-135
66	280	486	-206
67	237	473	-236
68	202	442	-240
69	300	401	-101
70	273	390	-117
71	389	373	16
72	646	391	255
73	732	476	256
74	359	564	-205
75	400	524	-124
76	253	496	-243
77	247	429	-182
78	249	374	-125
79	248	334	-86
80	536	310	226
81	249	378	-129
82	256	358	-102
83	254	347	-93
84	315	340	-25
85	208	352	-144
86	269	331	-62
87	203	331	-128
88	290	312	-22
89	608	320	288
90	357	415	-58
91	260	420	-160
92	189	394	-205
93	166	353	-187
94	262	317	-55
95	216	321	-105

TABLE II (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
96	224	320	-96
97	304	329	-25
98	493	366	127
99	234	451	-217
100	262	446	-184
101	466	445	21
102	367	495	-128
103	357	498	-141
104	414	489	-75
105	393	488	-95
106	243	474	-231
107	438	417	21
108	208	423	-215
109	404	366	38
110	948	379	569
111	735	544	191
112	1056	617	439
113	787	754	33
114	1339	776	563
115	453	926	-473
116	563	781	-218
117	475	678	-203
118	676	564	112
119	582	531	51

Mean Absolute Error = 161

Mean Error = -74.73

Error Variance = 3.92×10^4

TABLE III

GENERAL EXPONENTIAL SMOOTHING RESULTS
GROWING SINE WITH HARMONICS
CYCLE LENGTH 52 WEEKS

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
1	61	63	-2
2	64	73	-9
3	84	79	5
4	53	87	-34
5	87	83	4
6	116	88	28
7	119	102	17
8	108	114	-6
9	167	123	44
10	405	147	258
11	89	235	-146
12	85	219	-134
13	339	203	136
14	208	260	-52
15	80	269	-189
16	116	238	-122
17	141	220	-79
18	133	211	-78
19	125	200	-75
20	158	187	-29
21	164	185	-21
22	82	185	-103
23	203	161	42
24	139	175	-36
25	133	169	-36
26	198	163	35
27	151	177	-26
28	140	176	-36
29	175	174	1
30	359	183	176
31	280	244	36
32	1253	273	980
33	273	569	-296
34	150	535	-385
35	257	466	-209
36	559	435	124
37	258	491	-233
38	290	451	-161
39	382	426	-44
40	547	432	115
41	119	484	-365
42	125	408	-283
43	85	349	-264
44	54	294	-240
45	343	245	98

TABLE III (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
46	118	292	-174
47	96	275	-179
48	80	259	-179
49	262	247	15
50	131	292	-161
51	85	296	-211
52	177	287	-110
53	106	307	-201
54	435	302	133
55	145	389	-244
56	91	376	-285
57	126	346	-220
58	442	327	115
59	86	397	-311
60	121	352	-231
61	153	320	-167
62	327	302	25
63	191	335	-144
64	220	324	-104
65	336	324	12
66	280	358	-78
67	237	372	-135
68	202	373	-171
69	300	366	-66
70	273	389	-116
71	389	403	-14
72	646	449	197
73	732	559	173
74	359	672	-313
75	400	657	-257
76	253	649	-396
77	247	596	-349
78	249	545	-296
79	248	501	-253
80	536	462	74
81	249	509	-260
82	256	466	-210
83	254	431	-177
84	315	402	-87
85	208	394	-186
86	269	359	-90
87	203	347	-144
88	290	320	-30
89	608	323	285
90	357	416	-59
91	260	422	-162
92	189	398	-209
93	166	356	-190
94	262	313	-51
95	216	306	-90

TABLE III (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
96	224	287	-63
97	304	275	29
98	493	289	204
99	234	355	-121
100	262	338	-76
101	466	331	135
102	367	383	-16
103	357	398	-41
104	414	407	7
105	393	429	-36
106	243	440	-197
107	438	406	32
108	208	432	-224
109	404	389	15
110	948	407	541
111	735	576	159
112	1056	653	403
113	787	801	-14
114	1339	840	499
115	453	1016	-563
116	563	901	-338
117	475	825	-350
118	676	730	-54
119	582	702	-120

Mean Absolute Error = 154

Mean Error = -67.78

Error Variance = 3.94×10^4

TABLE IV

GENERAL EXPONENTIAL SMOOTHING RESULTS

LINEAR MODEL

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
1	61	11	50
2	64	42	22
3	84	65	19
4	53	88	-35
5	87	94	-7
6	116	109	7
7	119	128	-9
8	108	142	-34
9	167	148	19
10	405	170	235
11	89	257	-168
12	85	227	-142
13	339	201	138
14	208	258	-50
15	80	260	-180
16	116	222	-106
17	141	203	-62
18	133	196	-63
19	125	187	-62
20	158	177	-19
21	164	179	-15
22	82	182	-100
23	203	159	44
24	139	178	-39
25	133	173	-40
26	198	166	32
27	151	181	-30
28	140	177	-37
29	175	171	4
30	359	177	182
31	280	236	4
32	1253	257	996
33	273	568	-295
34	150	504	-355
35	257	418	-161
36	559	384	175
37	258	449	-191
38	290	406	-116
39	382	382	0
40	547	391	156
41	119	448	-329
42	125	360	-235
43	85	295	-210
44	54	233	-179
45	343	177	166
46	118	223	-105

TABLE IV (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
47	96	189	-93
48	80	157	-77
49	262	128	134
50	131	162	-31
51	85	148	-63
52	177	124	53
53	106	134	-28
54	435	120	315
55	145	210	-65
56	91	190	-99
57	126	159	-33
58	442	146	297
59	86	232	-146
60	121	190	-69
61	153	168	-15
62	327	161	166
63	191	209	-18
64	220	204	16
65	336	209	127
66	280	248	32
67	237	261	-24
68	202	257	-55
69	300	243	57
70	273	262	11
71	389	269	120
72	646	308	338
73	732	416	316
74	359	524	-165
75	400	491	-91
76	253	477	-224
77	247	422	-175
78	249	377	-128
79	248	343	-95
80	536	317	219
81	249	385	-136
82	256	349	-93
83	254	324	-70
84	315	304	11
85	208	307	-99
86	269	277	-8
87	203	273	-70
88	290	250	40
89	608	259	349
90	357	363	-6
91	260	365	-105
92	189	337	-148
93	166	294	-128
94	262	254	8
95	216	254	-38
96	224	239	-15

TABLE IV (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
97	304	231	73
98	493	249	244
99	234	321	-87
100	262	296	-34
101	466	286	180
102	367	340	27
103	357	351	6
104	414	357	57
105	393	377	16
106	243	387	-144
107	438	348	90
108	208	377	-169
109	404	330	74
110	948	353	595
111	735	536	199
112	1056	609	447
113	787	761	26
114	1339	794	545
115	453	984	-531
116	563	857	-294
117	475	793	-318
118	676	716	-40
119	582	718	-136

Mean Absolute Error = 128

Mean Error = -2.21

Error Variance = 3.67×10^4

TABLE V

GENERAL EXPONENTIAL SMOOTHING RESULTS
CONSTANT MODEL

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
1	61	0	61
2	64	35	29
3	84	47	37
4	53	61	-8
5	87	58	29
6	116	67	49
7	119	81	38
8	108	92	16
9	167	96	71
10	405	115	290
11	89	191	-102
12	85	164	-79
13	339	144	195
14	208	194	14
15	80	197	-117
16	116	168	-52
17	141	155	-14
18	133	151	-18
19	125	147	-22
20	158	141	17
21	164	145	19
22	82	150	-68
23	203	133	70
24	139	151	-12
25	133	148	-15
26	198	144	54
27	151	158	-7
28	140	156	-16
29	175	152	23
30	359	158	201
31	280	208	72
32	1253	226	1027
33	273	483	-210
34	150	430	-280
35	257	360	-103
36	559	334	225
37	258	391	-133
38	290	357	-67
39	382	341	41
40	547	351	196
41	119	400	-281
42	125	330	-205
43	85	279	-194
44	54	230	-176
45	343	186	157
46	118	225	-107

TABLE V (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
47	96	199	-103
48	80	173	-93
49	262	150	112
50	131	178	-47
51	85	166	-81
52	177	146	31
53	106	154	-48
54	435	142	293
55	145	215	-70
56	91	198	-107
57	126	171	-45
58	442	160	282
59	86	230	-144
60	121	194	-73
61	153	176	-23
62	327	170	157
63	191	209	-18
64	220	205	15
65	336	209	127
66	280	240	40
67	237	250	-13
68	202	247	-45
69	300	236	64
70	273	252	21
71	389	257	132
72	646	290	356
73	732	379	353
74	359	467	-108
75	400	440	-40
76	253	430	-177
77	247	386	-139
78	249	351	-102
79	248	326	-78
80	536	306	230
81	249	364	-115
82	256	335	-79
83	254	315	-61
84	315	300	15
85	208	304	-96
86	269	280	-11
87	203	277	-74
88	290	259	31
89	608	266	342
90	357	352	5
91	260	353	-93
92	189	330	-141
93	166	295	-129
94	262	262	0
95	216	262	-46
96	224	251	-27

TABLE V (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
97	304	244	60
98	493	259	234
99	234	318	-84
100	262	297	-35
101	466	288	178
102	367	332	35
103	357	341	16
104	414	345	69
105	393	362	31
106	243	370	-127
107	438	338	100
108	208	363	-155
109	404	324	80
110	948	344	604
111	735	495	240
112	1056	555	501
113	787	680	107
114	1339	707	632
115	453	865	-412
116	563	762	-199
117	475	712	-237
118	676	653	23
119	582	659	-77

Mean Absolute Error = 122

Mean Error = 20.26

Error Variance = 3.48×10^4

In this model the initial estimate of $a(0)$ was taken to be zero. Also, since the initial forecasts are biased by $1-h^t$, the method of finite exponential smoothing [Bessler and Zehna 1968] was used. With this procedure, the weight given the forecast error in determining the new value of $a(t)$ is $h(1-h^t)^{-1}$ rather than the constant h. This method increased the initial forecasts and as h^t vanished, the forecasts became identical with those using the constant h throughout. These results are in Table V.

B. FORECASTING WITH THE KALMAN FILTER

The Kalman Filter requires, in addition to the initial value of $\underset{\sim}{a}(0)$, the value of C_v^{-1} in order to forecast. In the models investigated, the matrix C_v was not known. However, since the basic model was

$$\underset{\sim}{C} = \underset{\sim}{B}\underset{\sim}{a} + \underset{\sim}{v}$$

and in forecasting one time period in advance, C_v is a scalar, it was decided that C_v could be estimated as the variance of the forecast error. Accordingly, the current estimate of the error variance was used as the value of C_v in the Kalman Filter equations.

The initial value of C_e was obtained from the equation

$$C_e = [B^T C_v^{-1} B]^{-1}.$$

This required at least two forecasts in order to estimate C_v. So that a better estimate could be made, C_e was initialized after ten time periods. The forecasts for these ten periods were obtained by using the initialized value of $\underset{\sim}{a}(0)$.

The forecasting procedure was:

Step 1; Initialize $\underset{\sim}{a}(10) = \underset{\sim}{a}(0)$.

Step 2; Obtain the forecasts for the first ten periods from

$$\hat{C}(t) = \underset{\sim}{b}^T(t)\underset{\sim}{a}(10).$$

Step 3; Compute the forecast errors, $e(t)$, for these ten periods and the mean error, $\bar{e}$.

Step 4; Obtain an estimate of C_v from

$$C_v = \frac{1}{10}\sum_{k=1}^{10} [e(k) - \bar{e}]^2.$$

Step 5; Initialize C_e by

$$C_e(10) = [B^T C_v^{-1} B]^{-1}.$$

Step 6; Obtain new forecast $\hat{C}(11)$ by

$$\hat{C}(11) = \underset{\sim}{b}^T(11)\underset{\sim}{a}(10).$$

Step 7; Compute forecast error $e(11)$ and the new error mean, $\bar{e}$.

Step 8; Obtain new estimate of C_v from

$$C_v = \frac{1}{11}\sum_{k=1}^{11} [e(k) - \bar{e}]^2 .$$

Step 9; Compute weighting vector $\underset{\sim}{K}$ by

$$\underset{\sim}{K} = C_e(10)\underset{\sim}{b}(11)[C_v + \underset{\sim}{b}^T(11)C_e(10)\underset{\sim}{b}(11)]^{-1}.$$

Step 10; Obtain new estimate $\underset{\sim}{a}(11)$ from

$$\underset{\sim}{a}(11) = \underset{\sim}{a}(10) + \underset{\sim}{K}e(11).$$

Step 11; Compute new value of $C_e(11)$ from

$$C_e(11) = C_e(10) - \underset{\sim}{K}\underset{\sim}{b}^T(11)C_e(10).$$

Step 12; Repeat Steps 6 through 11 with the appropriate time arguments for each successive forecast.

In general the value of C_v will be different from C_e. An exception is in the constant model where they are identical. In this model the computation of $\underset{\sim}{K}$ in Step 9 above involved the old estimate of the error variance as the value of C_e and C_v was the most recent estimate of the forecast error variance.

C. FORECASTING WITH LINEAR REGRESSION

The method of Least Squares was applied to the data using the linear model. The procedure employed was to fit the least squares line through the first twelve weekly loss figures. The coefficients of this line were used to obtain a forecast of losses for the thirteenth period. A new line was fitted to the thirteen observations and the new coefficients used for the next forecast. This procedure was continued through each succeeding time period.

TABLE VI

KALMAN FILTER RESULTS
GROWING SINE WITH HARMONICS
CYCLE LENGTH 16 WEEKS

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
1	61	47	14
2	64	28	36
3	84	19	65
4	53	37	16
5	87	76	11
6	116	115	1
7	119	129	-10
8	108	107	1
9	167	58	109
10	405	10	395
11	89	-7	96
12	85	384	-299
13	339	-191	530
14	208	1146	-938
15	80	-269	349
16	116	-590	706
17	141	189	-48
18	133	539	-406
19	125	468	-343
20	158	285	-127
21	164	189	-25
22	82	100	-18
23	203	-61	264
24	139	80	59
25	133	91	42
26	198	131	67
27	151	231	-80
28	140	255	-115
29	175	250	-75
30	359	235	124
31	280	278	2
32	1253	242	1011
33	273	603	-330
34	150	441	-291
35	257	272	-15
36	559	221	338
37	258	322	-64
38	290	275	15
39	382	236	146
40	547	226	321
41	119	273	-154
42	125	191	-66
43	35	183	-98
44	54	223	-169
45	343	288	55

TABLE VI (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
46	118	435	-317
47	96	453	-357
48	80	411	-331
49	262	339	-77
50	131	340	-209
51	85	330	-245
52	177	342	-165
53	106	389	-283
54	435	387	48
55	145	422	-277
56	91	321	-230
57	126	197	-71
58	442	118	324
59	86	178	-92
60	121	180	-59
61	153	220	-67
62	327	261	66
63	191	306	-115
64	220	275	-55
65	336	234	102
66	280	237	43
67	237	256	-19
68	202	293	-91
69	300	332	-32
70	273	374	-101
71	389	366	23
72	646	339	307
73	732	331	401
74	359	329	30
75	400	268	132
76	253	258	-5
77	247	253	-6
78	249	265	-16
79	248	272	-24
80	536	260	276
81	249	293	-44
82	256	265	-9
83	254	269	-15
84	315	312	3
85	208	393	-185
86	269	448	-179
87	203	484	-281
88	290	464	-174
89	608	419	189
90	357	404	-47
91	260	346	-86
92	189	301	-112
93	166	283	-117
94	262	287	-25
95	216	309	-93
96	224	301	-77

TABLE VI (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
97	304	273	31
98	493	257	236
99	234	290	-56
100	262	308	-46
101	466	363	103
102	367	459	-92
103	357	510	-153
104	414	513	-99
105	393	476	-83
106	243	406	-163
107	438	318	120
108	208	295	-87
109	404	274	130
110	948	310	638
111	735	409	326
112	1056	444	612
113	787	485	302
114	1339	463	876
115	453	503	-50
116	563	451	112
117	475	448	27
118	676	459	217
119	582	492	90

Mean Absolute Error = 166

Mean Error = 11.33

Error Variance = 6.28×10^4

TABLE VII

KALMAN FILTER RESULTS
GROWING SINE WITH HARMONICS
CYCLE LENGTH 32 WEEKS

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
1	61	60	1
2	64	63	1
3	84	63	21
4	53	61	-8
5	87	62	25
6	116	69	47
7	119	82	37
8	108	102	6
9	167	127	40
10	405	156	249
11	89	184	-95
12	85	-75	160
13	339	179	160
14	208	1155	-947
15	80	617	-537
16	116	969	-853
17	141	-362	503
18	133	162	-29
19	125	246	-121
20	158	183	-25
21	164	173	-9
22	82	117	-35
23	203	-102	305
24	139	215	-76
25	133	153	-20
26	198	165	33
27	151	345	-194
28	140	266	-126
29	175	227	-52
30	359	291	68
31	280	608	-328
32	1253	505	748
33	273	1782	-1509
34	150	1007	-857
35	257	258	-1
36	559	-12	571
37	258	198	60
38	290	32	258
39	382	21	361
40	547	144	403
41	119	384	-265
42	125	156	-31
43	85	50	35
44	54	-21	75
45	343	-67	410

TABLE VII (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
46	118	102	16
47	96	75	21
48	80	47	33
49	262	24	238
50	131	89	42
51	85	85	0
52	177	73	104
53	106	105	1
54	435	124	311
55	145	240	-95
56	91	278	-187
57	126	308	-182
58	442	351	91
59	86	466	-380
60	121	480	-359
61	153	492	-339
62	327	495	-168
63	191	518	-327
64	220	482	-262
65	336	436	-100
66	280	412	-132
67	237	367	-130
68	202	315	-113
69	300	263	37
70	273	244	29
71	389	227	162
72	646	245	401
73	732	324	408
74	359	411	-52
75	400	396	4
76	253	384	-131
77	247	332	-85
78	249	279	-30
79	248	229	19
80	536	184	352
81	249	206	43
82	256	168	88
83	254	142	112
84	315	127	188
85	208	134	74
86	269	133	136
87	203	156	47
88	290	177	113
89	608	222	386
90	357	325	32
91	260	371	-111
92	189	391	-202
93	166	391	-225
94	262	381	-119
95	216	379	-163

TABLE VII (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
96	224	361	-137
97	304	340	-36
98	493	332	161
99	234	358	-124
100	262	334	-72
101	466	322	144
102	367	347	20
103	357	356	1
104	414	365	49
105	393	382	11
106	243	393	-150
107	438	374	64
108	208	384	-176
109	404	351	53
110	948	348	600
111	735	419	316
112	1056	444	612
113	787	500	287
114	1339	504	835
115	453	565	-112
116	563	504	59
117	475	460	15
118	676	414	262
119	582	404	178

Mean Absolute Error = 189.5

Mean Error = 7.64

Error Variance = 9.05×10^4

TABLE VIII

KALMAN FILTER RESULTS
GROWING SINE WITH HARMONICS
CYCLE LENGTH 52 WEEKS

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
1	61	63	-2
2	64	74	-10
3	84	82	2
4	53	94	-41
5	87	-45	132
6	116	-79	195
7	119	14	105
8	108	200	-92
9	167	272	-105
10	405	357	48
11	89	738	-649
12	85	550	-465
13	339	90	249
14	208	112	96
15	80	-78	158
16	116	-322	438
17	141	-375	516
18	133	-391	524
19	125	-569	694
20	158	-1997	2155
21	164	8194	-8030
22	82	7445	-7363
23	203	5585	-5382
24	139	3774	-3635
25	133	2474	-2341
26	198	1713	-1515
27	151	1335	-1184
28	140	1058	-918
29	175	850	-675
30	359	706	-347
31	280	710	-430
32	1253	597	656
33	273	1214	-941
34	150	799	-649
35	257	398	-141
36	559	192	367
37	258	242	16
38	290	65	225
39	382	-47	429
40	547	-90	637
41	119	-75	194
42	125	-295	420
43	85	-537	622
44	54	-989	1043
45	343	-3151	3494

TABLE VIII (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
46	118	4022	-3904
47	96	1478	-1382
48	80	960	-880
49	262	755	-493
50	131	803	-672
51	85	751	-666
52	177	698	-521
53	106	727	-621
54	435	708	-273
55	145	877	-732
56	91	836	-745
57	126	777	-651
58	442	747	-305
59	86	865	-779
60	121	788	-667
61	153	740	-587
62	327	710	-383
63	191	747	-556
64	220	720	-500
65	336	703	-367
66	230	720	-440
67	237	710	-473
68	202	684	-482
69	300	649	-349
70	273	641	-368
71	389	623	-234
72	646	631	15
73	732	689	43
74	359	750	-391
75	400	724	-324
76	253	707	-454
77	247	663	-416
78	249	624	-375
79	248	590	-342
80	536	561	-25
81	249	583	-334
82	256	559	-303
83	254	540	-286
84	315	525	-210
85	208	522	-314
86	269	504	-235
87	203	498	-295
88	290	481	-191
89	608	479	129
90	357	527	-170
91	260	527	-267
92	189	509	-320
93	166	478	-312
94	262	443	-181
95	216	423	-207

TABLE VIII (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
96	224	395	-171
97	304	368	-64
98	493	351	142
99	234	359	-125
100	262	328	-66
101	466	302	164
102	367	303	64
103	357	291	66
104	414	280	134
105	393	278	115
106	243	276	-33
107	438	257	181
108	208	270	-62
109	404	255	149
110	948	271	677
111	735	358	377
112	1056	414	642
113	787	506	281
114	1339	560	779
115	453	675	-222
116	563	677	-114
117	475	690	-215
118	676	689	-13
119	582	710	-128

Mean Absolute Error = 651

Mean Error = -359.16

Error Variance = 17.47×10^5

TABLE IX

KALMAN FILTER RESULTS

LINEAR MODEL

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
1	61	-21	82
2	64	-6	70
3	84	10	74
4	53	26	27
5	87	41	46
6	116	57	59
7	119	73	46
8	108	88	20
9	167	104	63
10	405	120	285
11	89	135	-46
12	85	122	-37
13	339	127	212
14	208	185	23
15	80	209	-129
16	116	203	-87
17	141	203	-62
18	133	206	-73
19	125	206	-81
20	158	204	-46
21	164	208	-44
22	82	210	-128
23	203	200	3
24	139	209	-70
25	133	207	-74
26	198	203	-5
27	151	210	-59
28	140	209	-69
29	175	206	-31
30	359	208	151
31	280	234	46
32	1253	247	1006
33	273	286	-13
34	150	294	-144
35	257	298	-41
36	559	305	254
37	258	324	-66
38	290	330	-40
39	382	337	45
40	547	348	199
41	119	366	-247
42	125	364	-239
43	85	361	-276
44	54	357	-303
45	343	351	-8

TABLE IX (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
46	118	358	-240
47	96	355	-259
48	80	350	-270
49	262	344	-82
50	131	347	-216
51	85	343	-258
52	177	337	-160
53	106	335	-229
54	435	329	106
55	145	340	-195
56	91	336	-245
57	126	329	-203
58	442	324	118
59	86	335	-249
60	121	328	-207
61	153	322	-169
62	327	318	9
63	191	323	-132
64	220	321	-101
65	336	320	16
66	280	325	-45
67	237	327	-90
68	202	326	-124
69	300	324	-24
70	273	327	-54
71	389	328	61
72	646	334	312
73	732	353	379
74	359	374	-15
75	400	378	22
76	253	383	-130
77	247	382	-135
78	249	380	-131
79	248	378	-130
80	536	377	159
81	249	388	-139
82	256	386	-130
83	254	384	-130
84	315	382	-67
85	208	383	-175
86	269	380	-111
87	203	379	-176
88	290	375	-85
89	608	375	233
90	357	388	-31
91	260	390	-130
92	189	388	-199
93	166	384	-218
94	262	379	-117
95	216	377	-161

TABLE IX (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
96	224	374	-150
97	304	372	-68
98	493	372	121
99	234	380	-146
100	262	377	-115
101	466	376	90
102	367	382	-15
103	357	385	-28
104	414	387	27
105	393	391	2
106	243	394	-151
107	438	391	47
108	208	396	-188
109	404	392	12
110	948	395	553
111	735	416	319
112	1056	429	627
113	787	450	337
114	1339	462	877
115	453	485	-32
116	563	488	75
117	475	494	-19
118	676	497	179
119	582	505	77

Mean Absolute Error = 141

Mean Error = -15.33

Error Variance = 4.18×10^4

TABLE X

KALMAN FILTER RESULTS
CONSTANT MODEL

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
1	61	0	61
2	64	0	64
3	84	0	84
4	53	1	52
5	87	29	58
6	116	77	39
7	119	80	39
8	108	103	5
9	167	103	64
10	405	163	242
11	89	164	-75
12	85	117	-32
13	339	94	245
14	208	136	72
15	80	208	-128
16	116	208	-92
17	141	149	-8
18	133	142	-9
19	125	137	-12
20	158	131	27
21	164	157	7
22	82	158	-76
23	203	153	50
24	139	201	-62
25	133	198	-65
26	198	165	33
27	151	198	-47
28	140	196	-56
29	175	171	4
30	359	175	184
31	280	176	104
32	1253	260	993
33	273	267	6
34	150	273	-123
35	257	264	-7
36	559	258	301
37	256	271	-13
38	290	259	31
39	382	287	95
40	547	305	242
41	119	317	-198
42	125	247	-122
43	85	164	-79
44	54	112	-58
45	343	7	266

TABLE X (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
46	118	123	-5
47	96	118	-22
48	80	110	-30
49	62	96	166
50	131	141	-10
51	85	133	-48
52	177	120	57
53	106	176	-70
54	435	174	261
55	145	228	-83
56	91	166	-75
57	126	125	1
58	442	126	316
59	86	132	-46
60	121	90	31
61	153	120	33
62	327	140	187
63	191	141	50
64	220	191	29
65	336	194	142
66	280	198	82
67	237	270	-33
68	202	264	-62
69	300	242	58
70	273	299	-26
71	389	298	91
72	646	360	286
73	732	371	361
74	359	505	-146
75	400	401	-1
76	253	400	-147
77	247	391	-144
78	249	317	-68
79	248	266	-18
80	536	252	284
81	249	268	-19
82	256	250	6
83	254	255	-1
84	315	254	61
85	208	305	-97
86	269	304	-35
87	203	277	-74
88	290	254	36
89	608	290	318
90	357	290	67
91	260	356	-96
92	189	353	-164
93	166	302	-136
94	262	223	39
95	216	262	-46

TABLE X (CONTINUED

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
96	224	262	-38
97	304	241	63
98	493	296	197
99	234	302	-68
100	262	254	8
101	466	261	205
102	367	267	100
103	357	355	2
104	414	356	58
105	393	400	-7
106	243	398	-155
107	438	391	47
108	208	438	-230
109	404	437	-33
110	948	406	542
111	735	415	320
112	1056	660	396
113	787	813	-26
114	1339	788	551
115	453	798	-345
116	563	581	-18
117	475	563	-88
118	676	548	128
119	582	637	-55

Mean Absolute Error = 109

Mean Error = 41.33

Error Variance = 9.23 X 10^3

TABLE XI

LINEAR REGRESSION RESULTS

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
13	339	196	143
14	208	252	-44
15	80	256	-176
16	116	224	-108
17	141	208	-67
18	133	201	-68
19	125	193	-68
20	158	184	-26
21	164	184	-20
22	82	185	-103
23	203	170	33
24	139	179	-40
25	133	176	-43
26	198	172	26
27	151	178	-27
28	140	177	-37
29	175	174	1
30	359	177	182
31	280	203	77
32	1253	217	1036
33	273	350	-77
34	150	351	-201
35	257	337	-80
36	559	336	223
37	258	369	-111
38	290	366	-76
39	382	366	16
40	547	376	171
41	119	402	-283
42	125	383	-258
43	85	366	-281
44	54	347	-293
45	343	327	16
46	118	334	-216
47	96	320	-224
48	80	306	-226
49	262	291	-29
50	131	292	-161
51	85	283	-198
52	177	271	-94
53	106	266	-160
54	435	257	178
55	145	272	-127
56	91	265	-174
57	126	255	-129

TABLE XI (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
58	442	248	194
59	86	263	-177
60	121	253	-132
61	153	246	-93
62	327	241	86
63	191	248	-57
64	220	246	-26
65	336	246	90
66	280	253	27
67	237	256	-19
68	202	256	-54
69	300	255	45
70	273	259	14
71	389	261	128
72	646	270	376
73	732	293	439
74	359	319	40
75	400	323	77
76	253	330	-77
77	247	329	-82
78	249	327	-78
79	248	326	-78
80	536	324	212
81	249	337	-88
82	256	336	-80
83	254	334	-80
84	315	333	-18
85	208	334	-126
86	269	331	-62
87	203	330	-127
88	290	327	-37
89	608	327	281
90	357	342	15
91	260	345	-85
92	189	343	-154
93	166	339	-173
94	262	334	-72
95	216	333	-117
96	224	330	-106
97	304	327	-23
98	493	328	165
99	234	337	-103
100	262	334	-72
101	466	333	133
102	367	340	27
103	357	343	14

TABLE XI (CONTINUED)

WEEK	LOSSES	ESTIMATED LOSSES	ERROR
104	414	346	68
105	393	350	43
106	243	354	-111
107	438	352	86
108	208	357	-149
109	404	353	51
110	948	357	591
111	735	380	355
112	1056	395	661
113	787	421	366
114	1339	437	902
115	453	472	-19
116	563	474	89
117	475	480	-5
118	676	484	192
119	582	493	89

Mean Absolute Error = 139

Mean Error = 9.9

Error Variance = 4.53×10^4

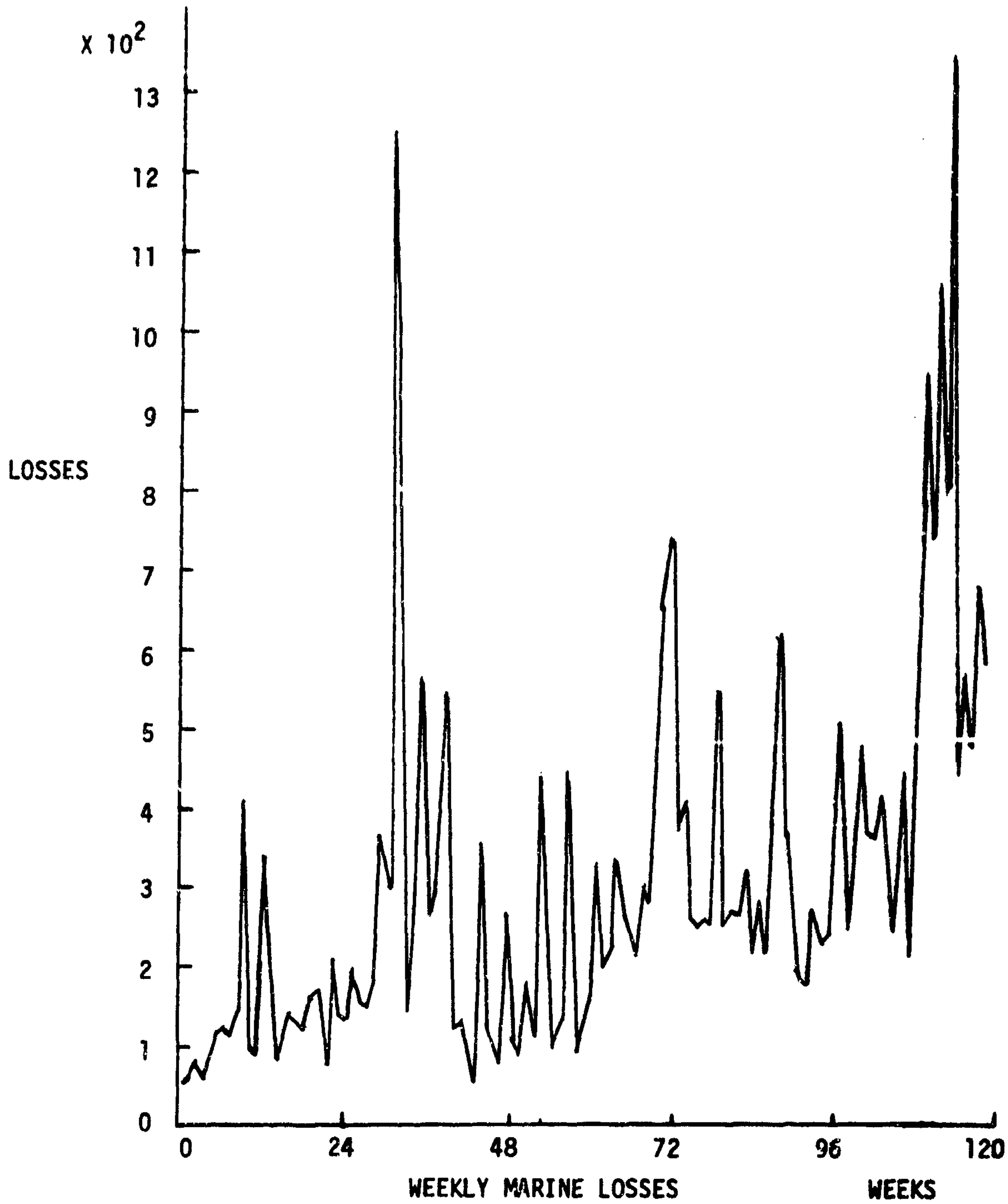
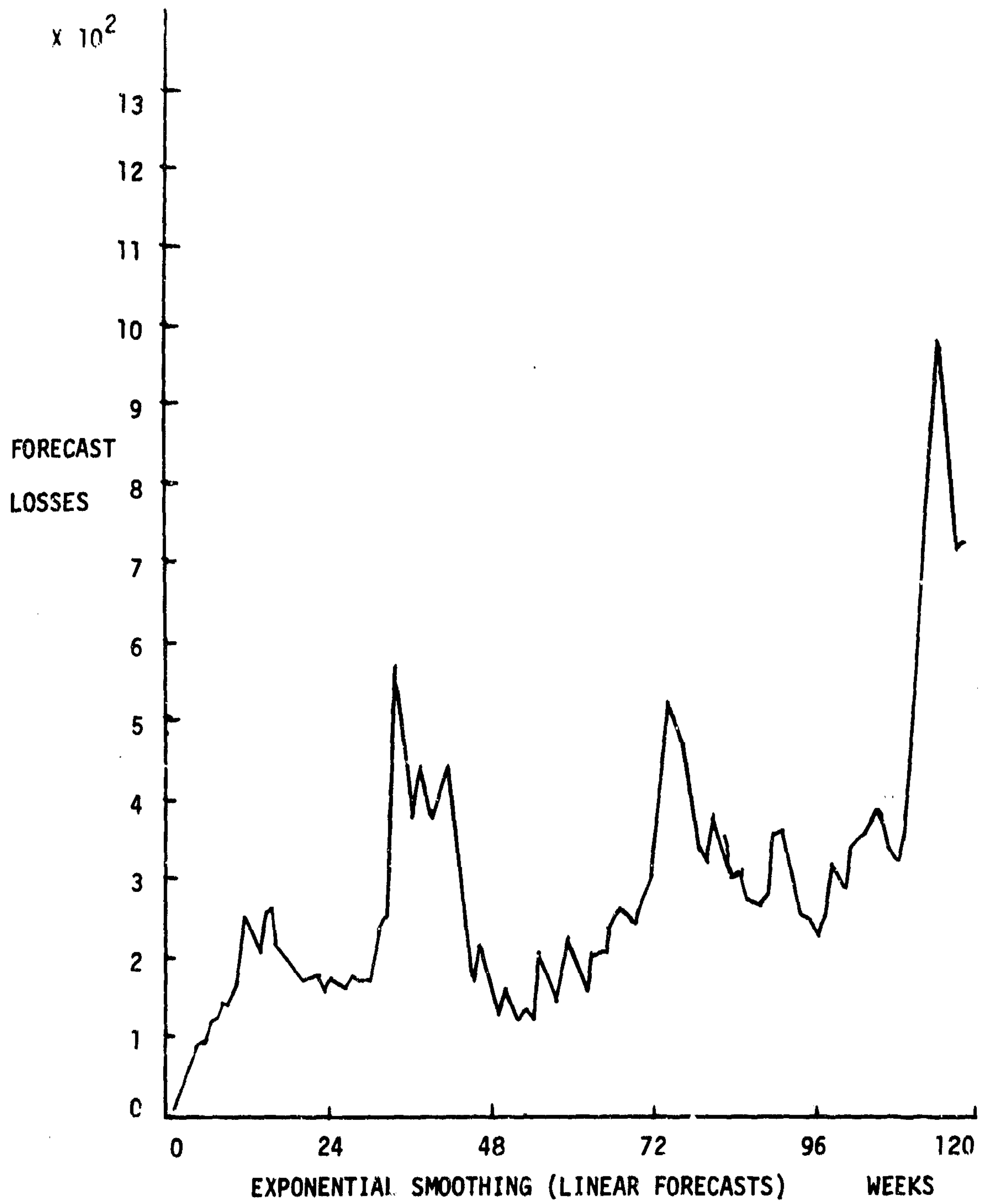

X 10^2
13
12
11
10
9
LOSSES
8
7
6
5
4
3
2
1
0
0
24
48
72
96
120
WEEKLY MARINE LOSSES
WEEKS

Figure 1



x 10^2
13
12
11
10
9
FORECAST LOSSES
8
7
6
5
4
3
2
1
0
0
24
48
72
96
120
EXPONENTIAL SMOOTHING (LINEAR FORECASTS)
WEEKS

Figure 2

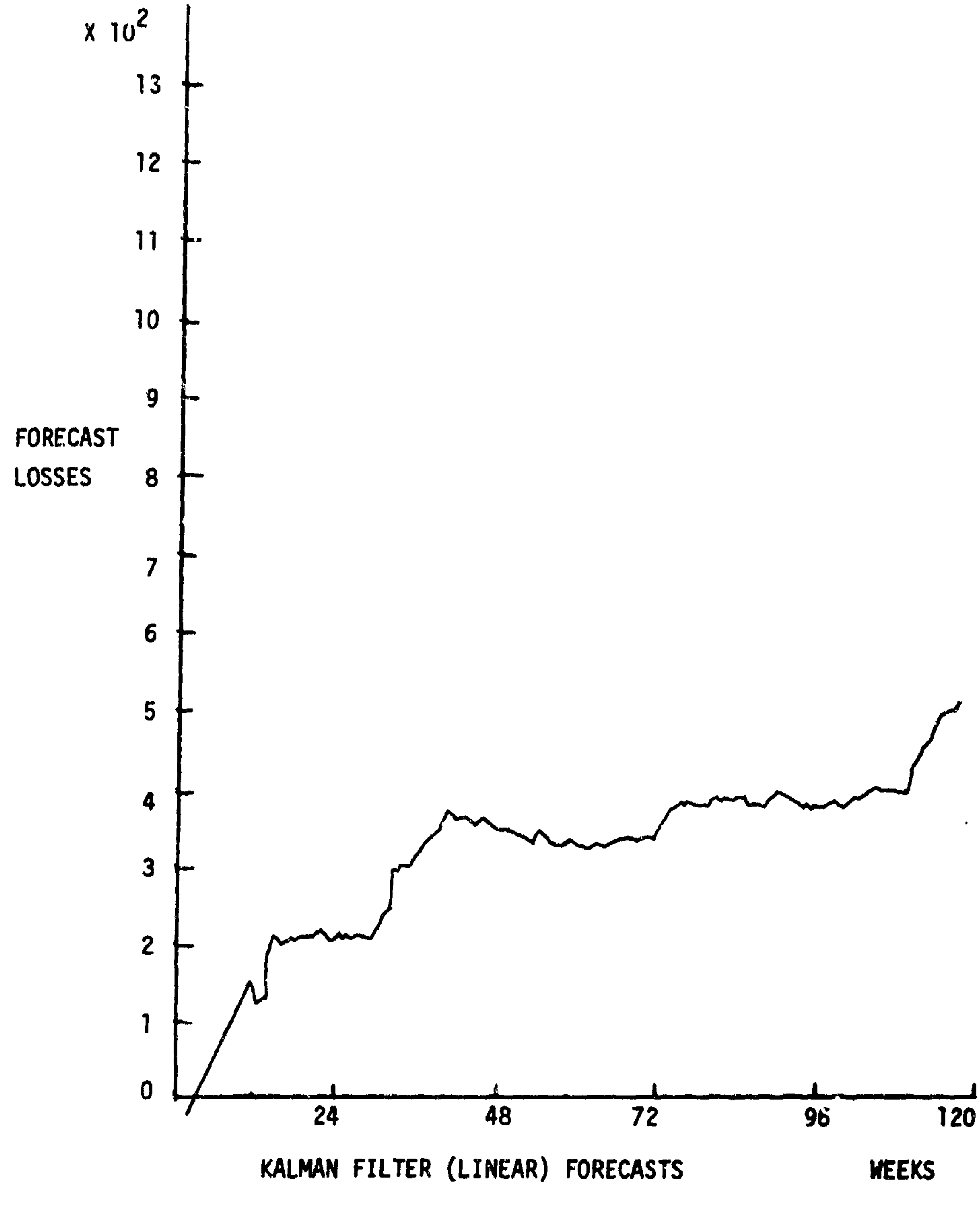
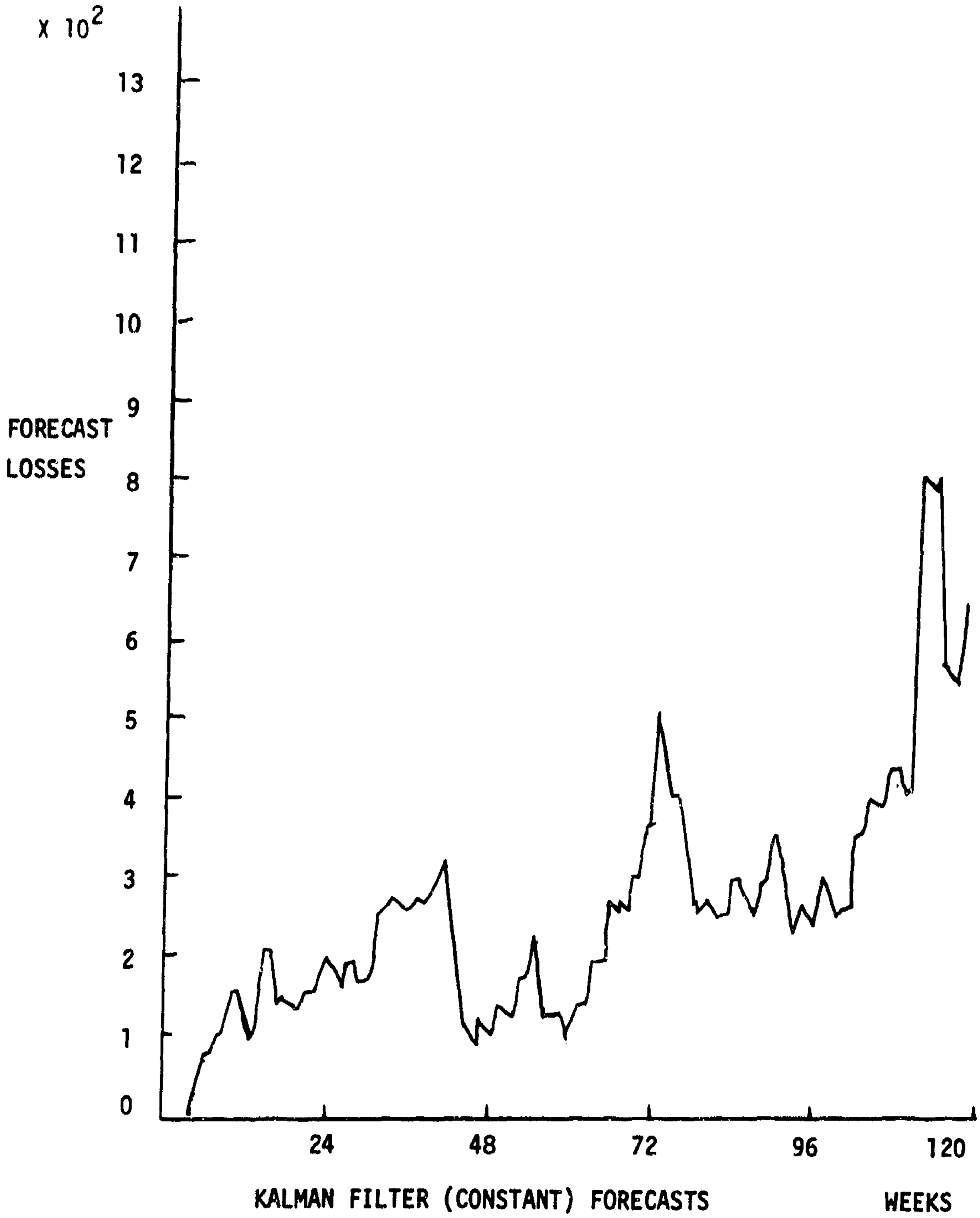

X 10^2
13
12
11
10
9
FORECAST
LOSSES
8
7
6
5
4
3
2
1
0
24
48
72
96
120
KALMAN FILTER (LINEAR) FORECASTS
WEEKS

Figure 3



X 10^2
13
12
11
10
9
FORECAST
LOSSES
8
7
6
5
4
3
2
1
0
24
48
72
96
120
KALMAN FILTER (CONSTANT) FORECASTS
WEEKS

Figure 4

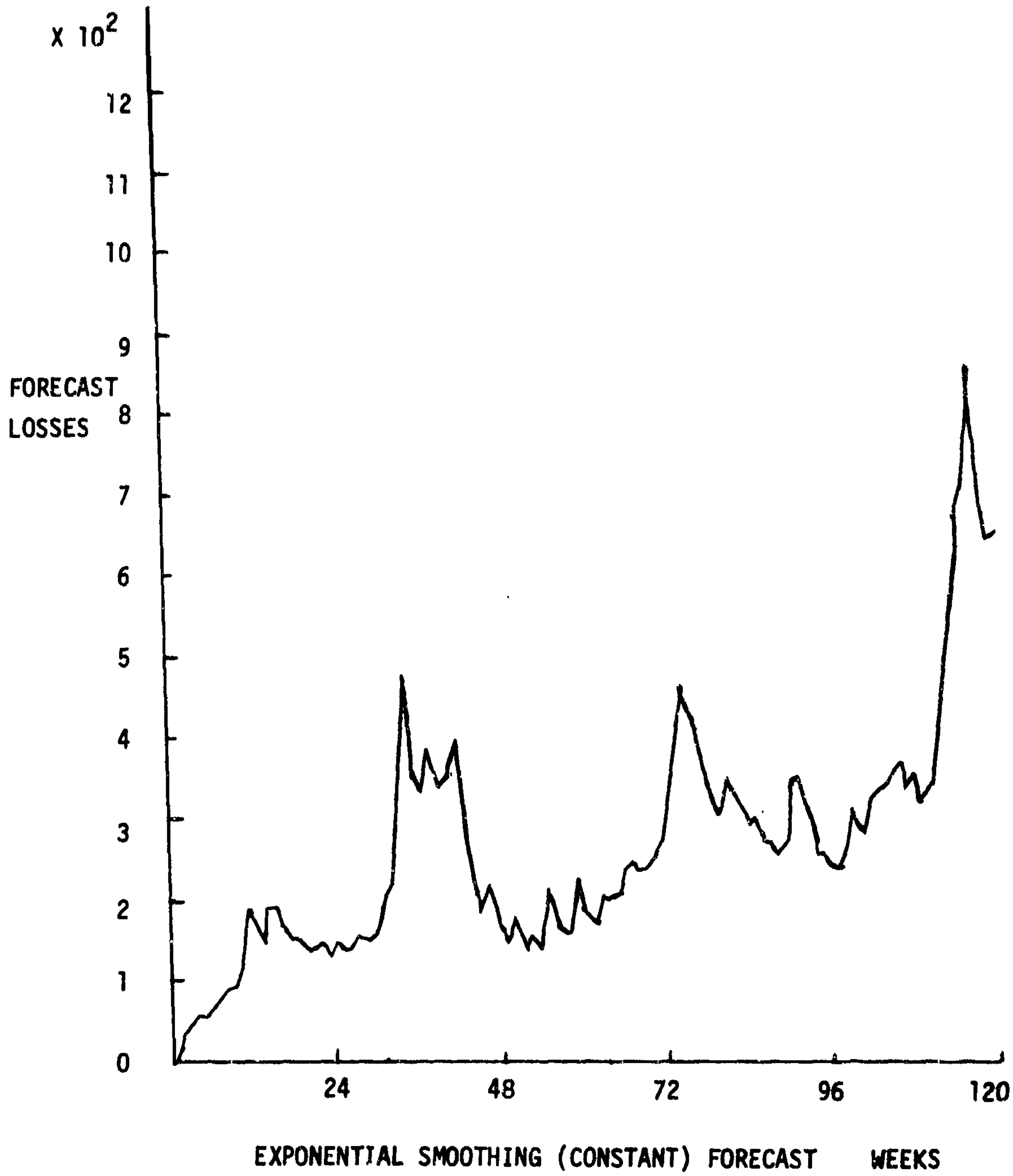

X 10^2
12
11
10
9
FORECAST
LOSSES
8
7
6
5
4
3
2
1
0
24
48
72
96
120
EXPONENTIAL SMOOTHING (CONSTANT) FORECAST
WEEKS

Figure 5

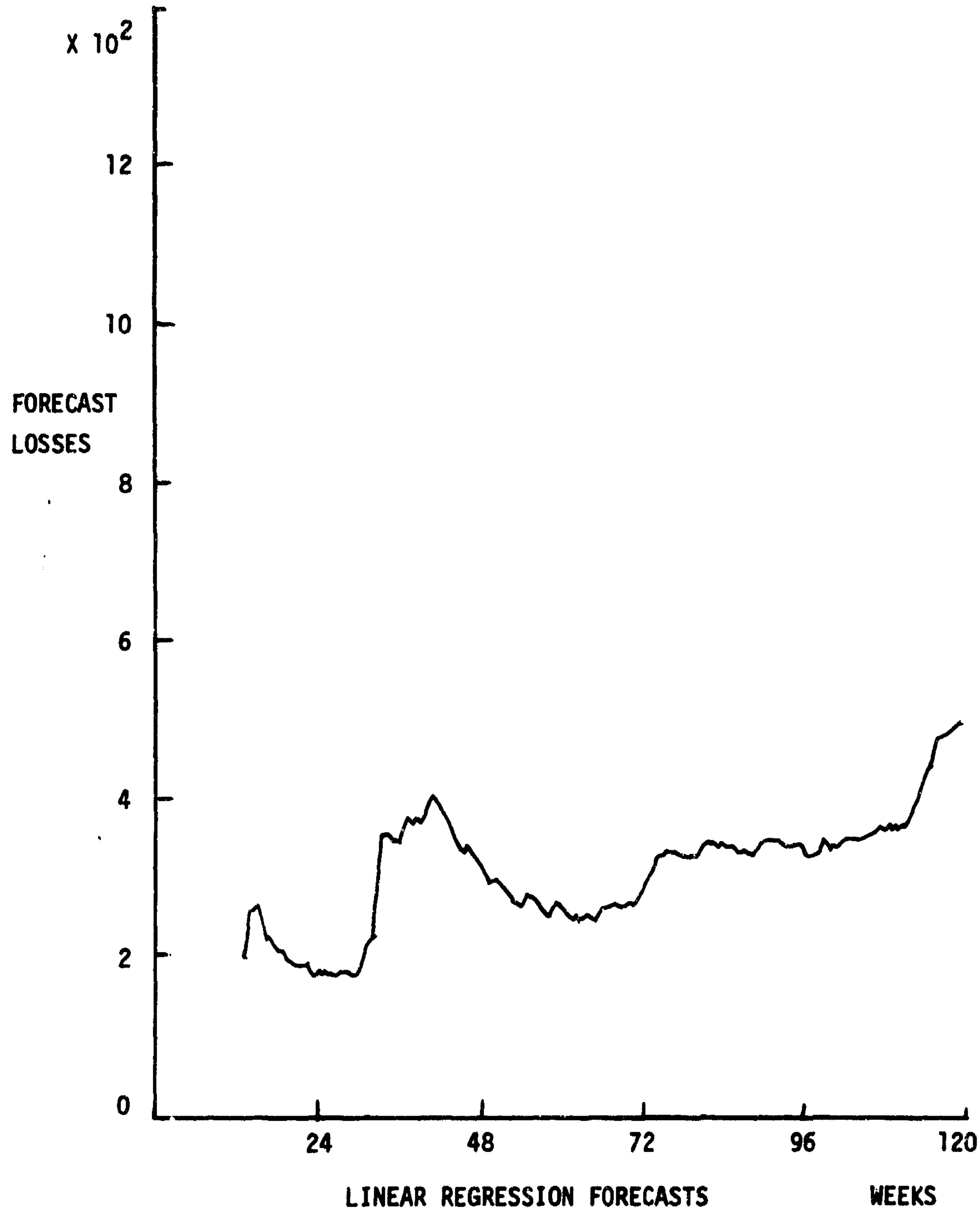

X 10^2
12
10
FORECAST
LOSSES
8
6
4
2
0
24
48
72
96
120
LINEAR REGRESSION FORECASTS
WEEKS

Figure 6

V. CONCLUSIONS

In the case of the linear model, it was shown in Chapter III that

$$E[\hat{C}(t)] = (a_1+a_2t)f^T(1)[I-L^T+\underset{\sim}{h}\ \underset{\sim}{f}^T(1)]^{-1}\ \underset{\sim}{h}$$

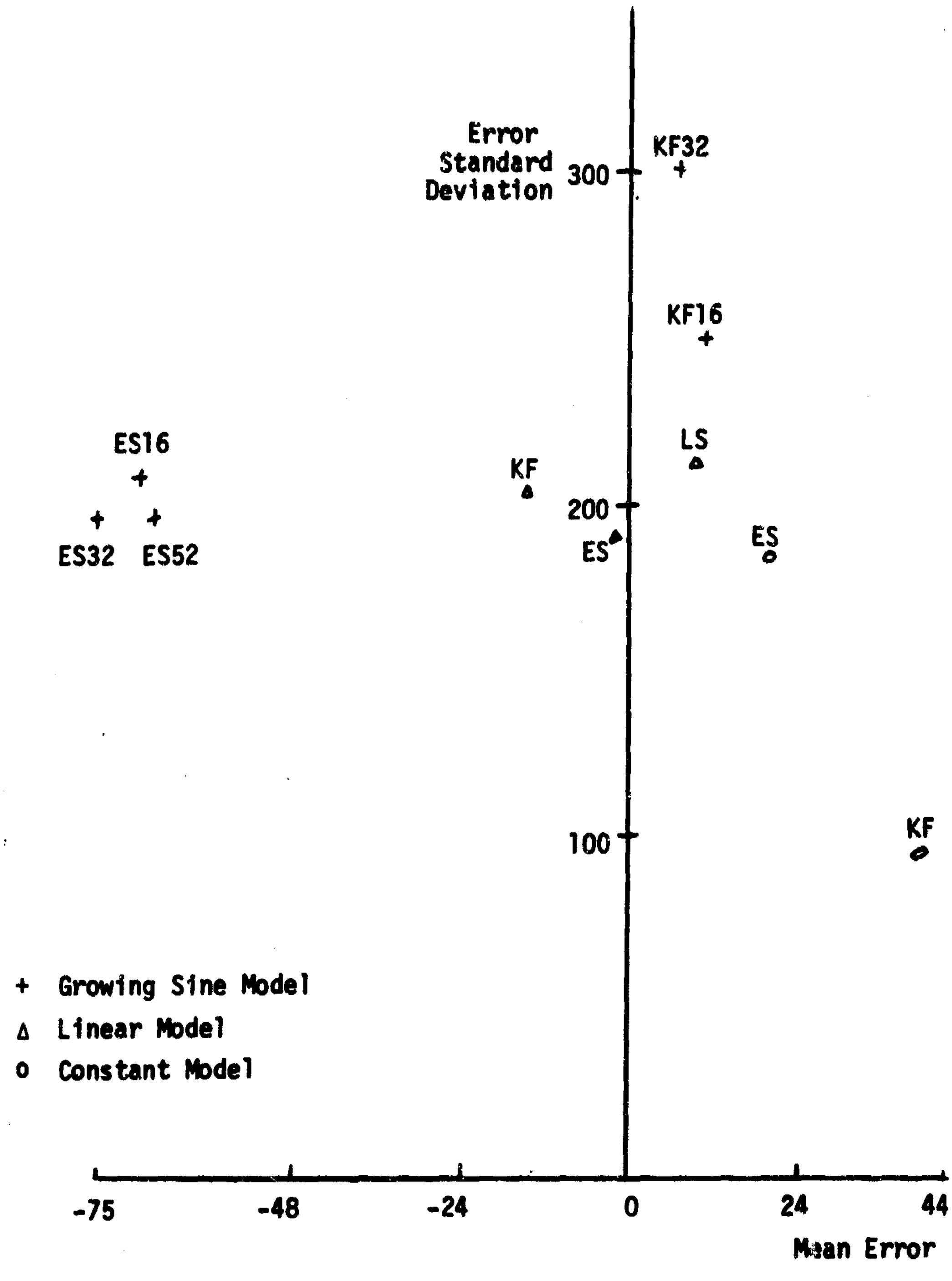
$$- a_2\ \underset{\sim}{f}^T(1)[I-L^T+\underset{\sim}{h}\ \underset{\sim}{f}^T(1)]^{-1}[L^T-\underset{\sim}{h}\ \underset{\sim}{f}^T(1)][I-L^T+\underset{\sim}{h}\ \underset{\sim}{f}^T(1)]^{-1}\ \underset{\sim}{h}$$

for large t and when the actual data was such that

$$E[C(t)] = a_1 + a_2t.$$

Then, $E[\hat{C}(t)] = a_1 + a_2t + a_2.$

This showed that when the assumed model was correct, the forecast was biased by the slope of the data line. In the case of the problem of forecasting the Marine losses, the least squares line through the one hundred and nineteen points had slope 3.41. This suggested that Exponential should have produced a forecast which is 3.41 more than the actual data, or an error of -3.41. The actual mean error was -2.21. Also, the Exponential Smoothing forecast would be above the linear regression forecast when the losses were increasing and below when they were decreasing. The results bear this out, although not around the time periods when the actual losses changed directions. For example, the losses increased during the time periods thirty-seven through forty. The Exponential Smoothing forecast with the linear model exceeded the linear regression forecasts during these periods. The losses then decreased during the periods forty-one through forty-four. The Exponential Smoothing forecast was above the linear regression forecast for the forty-first week but it then was below the linear regression forecasts through forty-four.



STANDARD DEVIATION OF ERRORS VS. MEAN ERROR

Figure 7

The results obtained with the Kalman Filter and the growing sine model were generally comparable with the results with Exponential Smoothing. This is evidenced in Figure 7. The exception was the sine model with a cycle length of fifty-two weeks. The Kalman Filter forecast for this model fluctuated badly for the initial forecasts. However, the last seventy weeks of the data produced a mean absolute error of -321.71 and a mean error of -208.71. The last forty weeks produced a mean absolute error of 219 and a mean error -24.08. When these results are compared with the overall mean absolute error of 651 and the mean error of -359.16, it appears that the forecasts were improving as the effects of the initial forecasts were dampening out. Thus, the model whould not be totally dismissed, especially since the Exponential Smoothing results with the cycle length of fifty-two weeks did not vary greatly from the results when a cycle length of sixteen and of thirty-two weeks was used. The Kalman Filter technique assumes that the forecast error variance is known. For the problem investigated, that variance was estimated, which was a variation on the Kalman Filter. If the true variance had been known, the results of all the models with which the Kalman Filter was used might have been improved.

With the models used in forecasting Marine losses, the simpler the model, the better the results. Captain O'Brien found that the linear model produced better results than the more complicated growing sine model for the prediction of monthly Marine losses [O'Brien 1968]. The same was true for the weekly forecasts. In addition, the even simpler constant model produced even better results. However, if the actual cycle period had been used rather than an estimate, the results of the Exponential Smoothing with the growing sine model might have been greatly improved. The period length can be determined as described in R.G. Brown's book[Brown 1963, p. 66-75].

In general, the estimated mean of the forecast errors was not zero. For example, the Kalman Filter with the constant model, on the average, forecasted forty-one fewer losses than actually occurred. With this information, a better policy would have been

$$\hat{C}'(t) = \hat{C}(t) + u,$$

where u is the mean forecast error, $\hat{C}'(t)$ is the estimated losses and $\hat{C}(t)$ is the forecasted losses from the method used. The variance of the forecast error will not be changed by this modification since $\hat{C}(t)$ would still be computed using $C(t-1) - \hat{C}(t-1)$. However, the mean error, where the error is

$$e'(t) = C(t) - \hat{C}'(t),$$

will be zero.

Finally, the results indicated, as evidenced in Figure 7, that even when an estimate of the forecast error covariance matrix C_v is used, the Kalman Filter results do not differ greatly from those of Exponential Smoothing. Since, except for the constant model, the Kalman Filter is more complicated than either the Exponential Smoothing or the Linear Regression methods, either of these latter methods should be used when C_v is unknown or when no computer facilities are available. When C_v is known and a computer is available, the Kalman Filter technique might provide better results than either of the other two methods.

Under the criterion chosen, the minimum mean absolute error, the Kalman Filter method using the constant model was the best forecasting method. This method could be easily used for forecasting Marine losses in Vietnam. The calculations for this constant model are relatively simple and can be accomplished with a calculator without the aid of a

computer. The calculation of the estimate of the forecast error variance is the most complicated portion of the constant model. The mean absolute error and the variance of the forecast errors achieved with this model compare favorably with the data mean of 294.6 and variance of the data of 5.21×10^4. Using this model would be more beneficial than merely using the data mean as the forecast of the losses for the next time period. Because of the difference between the two variances, one would be more certain of a minimum error with the Kalman Filter constant model.

GENERAL EXPONENTIAL SMOOTHING PROGRAM

This program computes successive weekly forecasts for the model

$$E[C(t)] = a_1 + a_2 t .$$

```
      DIMENSION A(2),FZERO(2),FONE(2),H(3,2),B(3),XL(2,2),
     *XEST(119),X(119),TIME(119),U(2),E(119),RELER(119),
     *Y(2),R(2)
      DATA B/0.86603,0.94868,0.97468/
      LM=119
         DO 91 J=1,119
   91 TIME(J)=J
      DO 3 J=1,2
      DO 3 I=1,2
    3 XL(I,J)=1.0
      XL(1,2)=0.0
      MMM=2
      READ(5,7)((H(I,J),J=1,MMM),I=1,3)
    7 FORMAT(4F13.5)
      READ(5,6) (X(I),I=1,119)
    6 FORMAT(11F6.0)
      FONE(1)=1.0
      FONE(2)=1.0
      DO 21 LI=1,MMM
      Y(LI)=0.0
      R(LI)=0.0
   21 CONTINUE
      DO 8 I=1,3
      ESUM=0.0
      A(1)=-145.E
      A(2)=62.7
C
C     CALCULATE ESTIMATE AND ERROR OF ESTIMATE
C
      DC 10 IT=1,LM
      S=C.0
      DC 40 LK=1,MMM
      S=S+A(LK)*FONE(LK)
   40 CONTINUE
      XEST(IT)=S
      E(IT)=X(IT)-XEST(IT)
      ESUM=ESUM+E(IT)
      EMEAN=ESUM/IT
      RELER(IT) = E(IT)/XEST(IT)
C
C     UPCATE COEFFICIENTS
C
      CALL GMPRD(A,XL,Y,1,2,2)
      DO 15 NL=1,MMM
      R(NL)=Y(NL)
   15 CONTINUE
      DO 20 J=1,MMM
      U(J) = 0.0
      U(J) = H(I,J)*E(IT)
   20 CONTINUE
      DC 25 M=1,MMM
      A(M) = R(M) + U(M)
   25 CCNTINUE
   10 CONTINUE
      WRITE(6,100)I,B(I)
  100 FCRMAT(1'1,23X,'CASE ',I1,' WEIGHTING CONSTANT, B = ',
```

```
     *F10.5,////)
      CALL XMOMEN(E,LM,XMEAN,SIGMA,DEV)
      SUMAB = 0.0
      DO 202 NN=1,LM
      SUMAB=SUMAB+ABS(E(NN))
  202 CONTINUE
      ABERR=SUMAB/LM
      WRITE(6,102)
  102 FORMAT(20X,'LOSSES',10X,'ESTIMATED LOSSES',10X,
     *'ERROR',7X,'RELATIVE ERROR',////)
      DC 35 MN=1,LM
      WRITE(6,103)   MN,X(MN),XEST(MN),E(MN),RELER(MN)
  103 FORMAT(3X,I3,13X,F7.2,10X,F8.2,16X,F8.2,14X,F6.3,/)
   35 CONTINUE
      WRITE(6,205)ABERR
  205 FORMAT(///,10X,'MEAN ABSOLUTE ERROR =   ',F4.0)
      WRITE(6,992)XMEAN,SIGMA,DEV
  992 FORMAT(/,10X,'MEAN ERROR = ',F8.2,//,10X,
     *'ERROR VARIANCE = ',E14.7,//,10X,'STANDARD DEVIATION =
     * ',F10.5)
      WRITE(6,201)
  201 FORMAT('1',38X,'PLOT OF TIME VS LOSSES AND ESTIMATED
     *LOSSES',///)
      CALL FLOTP(TIME,XEST,LM,1)
      CALL FLOTP(TIME,X,LM,3)
      WRITE(6,203)
  203 FORMAT('1',50X,'PLOT OF ERRORS VS.TIME',///)
      CALL FLOTP(TIME,E,LM,0)
    8 CONTINUE
      FONE(2)=FONE(2)+1.0
  177 CONTINUE
      STOP
      END

      SUBROUTINE XMOMEN(X,L,A,B,C)
      DIMENSION X(119)
      A=0.0
      B=0.0
      DO 10 I=1,L
   10 A = A+X(I)
      A = A/L
      DC 20 I=1,L
   20 B = B+(X(I)-A)**2
      B = B/L
      C = SQRT(B)
      RETURN
      END
```

KALMAN FILTER PROGRAM

This program computes successive weekly forecasts for the growing sinusoidal model with a cycle length of 32 weeks.

```
      REAL*8 C,CINV
      DIMENSION A(8),B(8),XL(8,8),E(119),X(119),XEST(119)
     *,RELER(119),U(8),CE(8,8),CK(8,8),C(8,8),CINV(8,8),
     *TIME(119),BB(25,8),Y(8),H(3)
      INTOBS=15
       L=119
      READ(5,1)(TIME(I),X(I),I=1,L)
    1 FORMAT(F5.C,58X,F6.0)
      N=8
      P=(2.0*3.141593)/32.0
  212 READ(5,12)(H(I),I=1,8)
   12 FORMAT(4F8.5)
       PP=2.0*P
      B(1)=1.0
      B(2)=1.0
      B(3)=SIN(P)
      B(4)=COS(P)
      B(5)=SIN(P)
      B(6)=B(4)
      B(7)=SIN(PP)
      B(8)=COS(PP)
       A(1)=-145.8/4.C
      A(2)=62.7/4.0
      A(3)=250.44/4.0
      A(4)=125.55/4.C
      A(5)=2.53/4.0
      A(6)=A(5)
      A(7)=-140.47/4.C
      A(8)=219.04/4.0
      DO 10 I=1,8
      DO 10 J=1,8
   10 XL(I,J)=0.0
      XL(1,1)=1.C
      XL(2,1)=1.0
      XL(2,2)=1.0
      XL(3,3)=COS(P)
      XL(3,4)=SIN(P)
      XL(4,3)=-XL(3,4)
      XL(4,4)=XL(3,3)
      XL(5,3)=XL(3,3)
      XL(5,4)=XL(3,4)
      XL(5,5)=XL(3,3)
      XL(5,6)=XL(3,4)
      XL(6,3)=-XL(3,4)
      XL(6,4)=XL(4,4)
      XL(6,5)=XL(6,3)
      XL(6,6)=XL(6,4)
      XL(7,7)=COS(PP)
      XL(7,8)=SIN(PP)
      XL(8,7)=-XL(7,8)
      XL(8,8)=XL(7,7)
      ESUM=0.0
      ABERR=0.0
      DO 19 J=1,INTOBS
      SUM=0.0
      DO 7 I=1,8
```

```
    7 SUM=SUM+A(I)*B(I)
      XEST(J)=SUM
      E(J)=X(J)-XEST(J)
      RELER(J)=E(J)/XEST(J)
      ESUM=ESUM+E(J)
      EMEAN=ESUM/J
      ABERR=ABERR+ABS(E(J))
      SUM=0.0
      DO 8 I=1,J
    8 SUM=SUM+(E(J)-EMEAN)**2
      CV=SUM/J
      DO 901 I=1,8
  901 BB(J,I)=B(I)
      CALL GMPRD(XL,B,Y,8,8,1)
      DO 4 M=1,8
    4 B(M)=Y(M)
   19 CONTINUE
      DO 18 I=1,8
      DO 18 K=1,8
      SUM=0.0
      DO 17 J=1,INTOBS
   17 SUM=SUM+BB(J,K)*BB(J,I)
   18 C(K,I)=SUM/CV
      CALL GAUSS3(8,C,C,CINV,KER,8)
      GO TO (2,3),KER
    3 WRITE(6,444)
  444 FORMAT(////,50X,'MATRIX SINGULAR',///)
      STOP
    2 DO 100 I=1,8
      DO 100 J=1,8
  100 CE(I,J)=CINV(I,J)
      INTOBS=INTOBS+1
      DO 14 IT=INTOBS,119
       SUM=0.0
      DO 15 I=1,8
   15 SUM=SUM+B(I)*A(I)
      XEST(IT)=SUM
      E(IT)=X(IT)-XEST(IT)
      RELER(IT)=E(IT)/XEST(IT)
      ABERR=ABERR+ABS(E(IT))
      ESUM=ESUM+E(IT)
      EMEAN=ESUM/IT
      SUM=0.0
      DO 16 I=1,IT
   16 SUM=SUM+(E(I)-EMEAN)**2
      CV=SUM/IT
      DO 303 I=1,8
      SUM=0.0
      DO 304 J=1,8
  304 SUM=SUM+CE(I,J)*B(J)
  303 U(I)=SUM
      SUM=0.0
      DO 305 I=1,8
  305 SUM=SUM+B(I)*U(I)
      DO 20 J=1,8
   20 CK(J)=U(J)/(CV+SUM)
C
C       UPDATE THE COEFFICIENTS
C
      DO 21 I=1,8
   21 A(I)=A(I)+CK(I)*E(IT)
      DO 22 I=1,8
      SUM=0.0
      DO 23 J=1,8
   23 SUM=SUM+B(J)*CE(J,I)
   22 U(I)=SUM
      DO 24 I=1,8
      DO 24 J=1,8
      CKCE(I,J)=CK(I)*U(J)
   24 CE(I,J)=CE(I,J)-CKCE(I,J)
      CALL GMPRD(XL,B,Y,8,8,1)
      DO 13 J=1,8
```

```
13 B(J)=Y(J)
14 CONTINUE
   WRITE(6,25)
25 FCRMAT('1',20X,'LCSSES',10X,'ESTIMATED LOSSES',10X,
  *'RELATIVE ERROR',///)
  *'ERROR',7X,'RELATIVE ERROR',///)
   ABERR=ABERR/L
   DEV=SQRT(CV)
   DC 26 MN=1,L
26 WRITE(6,27)MN,X(MN),XEST(MN),E(MN),RELER(MN)
27 FORMAT(3X,I3,14X,F6.0,12X,F7.2,18X,F7.2,14X,F6.3,/)
   WRITE(6,28) ABERR,EMEAN,CV,DEV
28 FORMAT(10X,'MEAN ABSOLUTE ERROR = ',F8.2,//,
  *'MEAN ERROR = ',F8.2,//,10X,'ERROR VARIANCE = ',E15.7,
  *//,10X,'ERROR STANCARD DEVIATION = 'E15.7)
   WRITE(6,29)
29 FORMAT('1',38X,'PLCT CF TIME VS. LOSSES ANC ESTIMATED'
  *' LOSSES',//)
   CALL PLOTP(TIME,XEST,L,1)
   CALL PLOTP(TIME,X,L,3)
   WRITE(6,30)
30 FORMAT('1',50X,'PLCT CF ERRORS VS. TIME',//)
   CALL PLOTP(TIME,E,L,0)
214 STCP
   END
```

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13. ABSTRACT

This paper investigates the application of the Kalman Filter and the General Exponential Smoothing techniques of forecasting. Both methods are derived and the similarities and differences between them are discussed. The two techniques are then applied to the practical problem of predicting weekly losses suffered by the U. S. Marine Corps units in the I Corps Tactical Zone in the Republic of Vietnam. The mean absolute error of the prediction is used as the criterion for shoosing the better of the two methods. Results are given for both techniques as well as for the method of linear regression. In general the Kalman Filter provides the smallest mean absolute error for the three mathematical models; linear, growing sine with harmonics and frequency of sixteen, thirty-two, and fifty-two weeks, and a constant model.

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Exponential Smoothing						
Forecasting Marine Losses						